Math 317 Week 03: Power Series

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References

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1. INTRODUCTION

DEFINITION 1. *A power series is a particular type of infinite series of functions:*

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{1}$$

which is often denoted in the compact form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{2}$$

Remark 2. Intuitively, a power series is simply a polynomial whose degree is infinite.

Remark 3. It is crucial to understand that the index n in the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is only a "place holder". Therefore we can replace n by any other symbol:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \ \sum_{m=0}^{\infty} a_m (x - x_0)^m, \ \sum_{k=0}^{\infty} a_k (x - x_0)^k, \ \text{or even} \sum_{l=2}^{\infty} a_{l-2} (x - x_0)^{l-2} \tag{3}$$

all denote the **same** power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{4}$$

However, they are not the same as

$$\sum_{n=2}^{\infty} a_n (x - x_0)^n \text{ or } \sum_{k=0}^{\infty} a_k (x - x_0)^{k+1} \tag{5}$$

as the former starts from a different term, and the latter has a different relation between the subscript and the power.

The motivation of the study of power series are two-fold.

1.1. Motivation 1: Taylor expansion

The study of power series can be viewed as the inverse problem for the study of Taylor expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \tag{6}$$

In essence, Taylor expansion is the following relation

$$f(x) = \text{a power series} = \text{a polynomial of degree infinity}. \tag{7}$$

We naturally ask: If

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{8}$$

is the Taylor expansion of $f(x)$ at x_0, does it hold that

$$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots + a_n (x - x_0)^n + \cdots ? \tag{9}$$

$$f'(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots + a_n (x - x_0)^n + \cdots ? \tag{10}$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b a_0\,\mathrm{d}x + \int_a^b a_1 (x - x_0)\,\mathrm{d}x + \cdots ? \tag{11}$$

Example 4. The Taylor expansion of $\ln(1 + x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \tag{12}$$

But can we conclude

$$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \tag{13}$$

1.2. Motivation 2: Power series method for ordinary differential equations

More practically, the solution to ordinary differential equations can be naturally obtained as a power series.

Example 5. Solve Airy's equation

$$y'' = x\,y. \tag{14}$$

The method proceeds as follows.

1. Write power series expansion for y:

$$y = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n. \tag{15}$$

   Note that here we are taking $x_0 = 0$ which is usually the first choice.

2. Substitute into the equation:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)'' = x \left(\sum_{n=0}^{\infty} a_n x^n \right). \tag{16}$$

3. Simplify both sides (Red = means "the operation needs justification")

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)'' = \sum_{n=0}^{\infty} (a_n x^n)'' = \sum_{n=2}^{\infty} a_n n (n - 1) x^{n-2} \tag{17}$$

$$x \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{18}$$

   Note that the $n = 0$ term a_0 and the $n = 1$ terms $a_1 x$ disappear when taking two derivatives.

So the equation becomes

$$\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} = \sum_{n=0}^{\infty} a_n\, x^{n+1}. \tag{19}$$

Remark. Keep in mind that $\sum \cdots$ is just a **notation**, a **short hand**. So $\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2}$ is just a notation for the "real" power series

$$2\, a_2 + 6\, a_3\, x + \cdots \tag{20}$$

and $\sum_{n=0}^{\infty} a_n\, x^{n+1}$ is just a notation for $a_0\, x + a_1\, x^2 + \cdots$. In particular, we are free to change either or both n to other symbols:

$$\sum_{m=2}^{\infty} a_m\, m\, (m-1)\, x^{m-2} = \sum_{n=0}^{\infty} a_n\, x^{n+1}, \tag{21}$$

$$\sum_{m=2}^{\infty} a_m\, m\, (m-1)\, x^{m-2} = \sum_{l=0}^{\infty} a_l\, x^{l+1}, \tag{22}$$

whatever... They all mean the **same** equation.

4. Shift indices. Note that if we do not use the short hands and just write what the equation really means:

$$2\, a_2 + 6\, a_3\, x + \cdots = a_0\, x + a_1\, x^2 + \cdots \tag{23}$$

we can immediately conclude

$$2\, a_2 = 0, \qquad 6\, a_3 = a_1, \tag{24}$$

and so on. This is not satisfactory though[1] as we won't be able to get a "universal" relation between a_n's. To get such relation we have to analyze

$$\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} = \sum_{n=0}^{\infty} a_n\, x^{n+1}. \tag{25}$$

We try to do the same thing – Equate the terms on both sides with the same power of x. To do this efficiently we need to "shift indices", that is introducing one or more new indices so that the generic terms (currently $a_n\, n\, (n-1)\, x^{n-2}$ and $a_n\, x^{n+1}$) has the same power of x.

- Shift left hand side. We try to re-write

$$\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} = \sum \cdots x^k. \tag{26}$$

It is clear that we should let the new index, k, be related to the old one, n, through

$$k = n - 2. \tag{27}$$

1. For a certain type of problems this is satisfactory. We will see.

Now replace every n by $k+2$:

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} = \sum_{k+2=2}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k \tag{28}$$

and simplify to

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k; \tag{29}$$

- Similarly for the right hand side,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{30}$$

5. Balance the equation. The equation now becomes

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{31}$$

Equating terms with same x^k on both sides, we reach

$$(k=0)\colon 2\,a_2 \;=\; 0; \tag{32}$$

$$(k\geqslant 1)\colon a_{k+2}\,(k+2)\,(k+1) \;=\; a_{k-1}; \tag{33}$$

This is called the "recurrence relation".

Remark. Note that when shifting indices, we use "k" as the new index for both sides. This is just for convenience since at the end of the day we would like to pick terms with the same x^k from both sides. We can also use difference symbols and reach:

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k = \sum_{l=1}^{\infty} a_{l-1}\, x^l. \tag{34}$$

This is the same equation as

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k = \sum_{k=1}^{\infty} a_{k-1}\, x^k. \tag{35}$$

The only difference is that we cannot proceed before renaming l by k!

Remark. Also, usually the index-shifting is done somewhat implicitly, that is we do not explicitly introduce any new symbols. In the future we will just replace every n by $n+2$ and obtain

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\,x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,x^n. \tag{36}$$

6. Write down the solution.

From the recurrence relation we can get:

$$a_{k+2} = \frac{a_{k-1}}{(k+2)\,(k+1)} = \frac{a_{k-4}}{(k+2)\,(k+1)\,(k-1)\,(k-2)} = \cdots \tag{37}$$

which finally can be written as

$$a_k = \begin{cases} \dfrac{a_0}{\Pi_{i=1}^{l}(3\,i)\,(3\,i-1)} & k = 3\,l \\ \dfrac{a_1}{\Pi_{i=1}^{l}(3\,i+1)\,(3\,i)} & k = 3\,l+1 \\ 0 & k = 3\,l+2 \end{cases} . \tag{38}$$

So the final answer is

$$y(x) = \sum_{k=0}^{\infty} a_k\,x^k \tag{39}$$

with a_k given by the above.

Exercise 1. Find out all $x \in \mathbb{R}$ such that $y(x)$ is well-defined by (39). Is $y(x)$ continuous? Is it differentiable? Obtain an infinite series representation for $y'(x)$.

Remark 6. Airy's equation looks very simple but its general solution cannot be written in "closed form". On the other hand, these solutions are very useful in practice that they were given a name and became one class of the so-called "Special functions".

Exercise 2. Solve the equation $y' = 3\,y$. Try to obtain the solution in closed form.

Exercise 3. Find $y(x)$ satisfying the equation $y'' = 4\,y$ as well as $y(0) = 0, y'(0) = 1$. Try to obtain the solution in closed form.

1.3. Theoretical issues to be settled

In the above solution we have done many dubious operations and thus left many theoretical gaps open. For example

$$\left(\sum_{n=0}^{\infty} a_n\,x^n\right)'' = \sum_{n=0}^{\infty} (a_n\,x^n)''. \tag{40}$$

Why can we do this? Does this hold for all x? If not, how do we tell for which x the above holds?

Turns out, all these gaps are filled as long as we restrict ourselves to $|x| < \rho$, where $0 \leqslant \rho \leqslant \infty$[2] is a certain number, determined by the coefficients $a_0, a_1, \ldots$, called "radius of convergence".

In short, given a power series

$$\sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \tag{41}$$

let ρ be its radius of convergence. Then in $|x - x_0| < \rho$ we can treat it as if it's a polynomial: Termwise differentiation, termwise integration, re-arrangement of terms, etc. All OK.

On the other hand, for those x such that $|x - x_0| > \rho$, the infinite sum

$$\sum_{n=0}^{\infty} a_n\,(x - x_0)^n \tag{42}$$

diverges. In other words, for those x this sum does not represent a function at all.

2. Notice it's $\leqslant$, not $<$!

2. Properties of Power Series

2.1. Radius of convergence

THEOREM 7. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series.*

a) If the power series converges for $x_1 \in \mathbb{R}$, then it converges absolutely for every x satisfying $|x - x_0| < |x_1 - x_0|$. Furthermore the convergence is uniform on $[x_0 - r, x_0 + r]$ for every $0 < r < |x_1 - x_0|$;

b) If the power series diverges for $x_2 \in \mathbb{R}$, then it diverges for every $x \in \mathbb{R}$ satisfying $|x - x_0| > |x_2 - x_0|$.

Exercise 4. Show that the strict inequalities ($|x - x_0| < |x_1 - x_0|$, $|x - x_0| > |x_2 - x_0|$) cannot be replaced by $\leqslant$, $\geqslant$. (Hint:[3])

Proof. We prove a) and leave b) as an exercise.

By the root test, we see that the convergence of $\sum_{n=0}^{\infty} a_n (x_1 - x_0)^n$ implies

$$\limsup_{n \to \infty} |a_n (x_1 - x_0)^n|^{1/n} \leqslant 1 \Longrightarrow \left(\limsup_{n \to \infty} |a_n|^{1/n} \right) |x_1 - x_0| \leqslant 1. \tag{43}$$

Now if $|x - x_0| < |x_1 - x_0|$ we have

$$\left(\limsup_{n \to \infty} |a_n|^{1/n} \right) |x - x_0| < 1 \tag{44}$$

and absolute convergence follows from the root test.

Now for any $0 < r < |x_1 - x_0|$, take x_3 such that $r < |x_3 - x_0| < |x_1 - x_0|$. Then for any $x \in [x_0 - r, x_0 + r]$, we have

$$\begin{aligned} a_n |x - x_0|^n + \cdots + a_m |x - x_0|^m &= a_n |x_3 - x_0|^n \cdot \left| \frac{x - x_0}{x_3 - x_0} \right|^n + \cdots + a_m |x_3 - x_0|^m \cdot \left| \frac{x - x_0}{x_3 - x_0} \right|^m \\ &\leqslant \left| \frac{x - x_0}{x_3 - x_0} \right|^n \cdot [a_n |x_3 - x_0|^n + \cdots + a_m |x_3 - x_0|^m] \end{aligned} \tag{45}$$

from which uniform convergence easily follows. Left as exercise. □

Exercise 5. Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $\sum_{n=0}^{\infty} a_n y^n$ be two power series. Prove that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges

3. Consider $\sum_{n=0}^{\infty} (-x)^n$.

at x if and only if $\sum_{n=0}^{\infty} a_n y^n$ converges at $y = x - x_0$. (Hint:[4])

Exercise 6. Finish the proof of a). (Hint:[5])

Exercise 7. Prove b).

Exercise 8. Prove the above theorem using the comparison test instead of the root test.

COROLLARY 8. (RADIUS OF CONVERGENCE) *Let* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *be a power series. Then there is* $R \in [0, \infty]$ *(note that* ∞ *is included!) such that*

- *If* $|x - x_0| < R$*, the series is absolutely convergent. Furthermore for any* $0 < r < \rho$*, the series is uniformly convergent on* $[x_0 - r, x_0 + r]$*;*
- *If* $|x - x_0| > R$*, the series is divergent.*
- *If* $|x - x_0| = R$*, the series may or may not be convergent.*

This R is called the "radius of convergence" for the power series. Furthermore we have

$$R^{-1} = \limsup_{n \to \infty} |a_n|^{1/n}. \tag{46}$$

Exercise 9. Prove the corollary.

Exercise 10. Find the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n!}, \qquad \sum_{n=1}^{\infty} (-x)^n, \qquad \sum_{n=1}^{\infty} (n!)\, x^n. \tag{47}$$

(Ans:[6])

2.2. Continuity, differentiation, integration

Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series with radius of convergence $R \in [0, \infty]$. Then $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is defined at least on $(x_0 - R, x_0 + R)$.[7]

THEOREM 9. (CONTINUITY) $f(x)$ *is continuous on* $(x_0 - R, x_0 + R)$.

Exercise 11. Prove the above theorem. (Hint:[8])

4. Root test.
5. By (44) we have $a_n r^n$ uniformly bounded.
6. $1, \infty, 1, 0$.
7. When the power series converges at one or both the end points, the situation is subtle and will be discussed later.
8. For any $x \in (x_0 - R, x_0 + R)$, there is $r < R$ such that $|x - x_0| < r$. But the convergence on $[x_0 - r, x_0 + r]$ is uniform.

THEOREM 10. (DIFFERENTIABILITY) *$f(x)$ is differentiable on $(x_0 - R, x_0 + R)$. Furthermore*

$$f'(x) = \sum_{n=0}^{\infty} n\, a_n\, (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_n\, (x - x_0)^n. \tag{48}$$

Exercise 12. Prove the above theorem. (Hint:[9])

COROLLARY 11. *$f(x)$ is infinitely continuously differentiable on $(x_0 - R, x_0 + R)$. Furthermore*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n\,(n-1)\cdots(n-k+1)\, a_n\, (x - x_0)^{n-k}. \tag{49}$$

Exercise 13. Prove the corollary.

THEOREM 12. (INTEGRABILITY) *$f(x)$ is integrable on $[a, b] \subset (x_0 - R, x_0 + R)$. Furthermore for any $a \in (x_0 - R, x_0 + R)$, there holds*

$$\int_{x_0}^{a} f(x)\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}. \tag{50}$$

Example 13. Calculate $\sum_{n=1}^{\infty} n\, x^n$.

Solution. It is easy to see that the radius of convergence is $R = 1$, and the series diverges at $x = -1, 1$. Thus in the following we only consider $x \in (-1, 1)$.

Recall that for such x,

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \Longrightarrow \sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{(1-x)^2}. \tag{51}$$

Therefore

$$\sum_{n=1}^{\infty} n\, x^n = \frac{x}{(1-x)^2} \tag{52}$$

for all $x \in (-1, 1)$.

Now taking $x = 1/2$, $1/3$ we obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2, \qquad \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}. \tag{53}$$

Exercise 14. Prove

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x); \qquad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} x^{2n+1}. \tag{54}$$

for $x \in (-1, 1)$. (Hint:[10])

9. Observe that $\sum_{n=0}^{\infty} (n+1)\, a_n\, (x - x_0)^n$ has the same radius of convergence.

10. Differentiate term by term.

3. Power Series Representation of Functions

In this section we settle the theoretical issues raised in the two motivating applications of Introduction.

3.1. Function and its Taylor expansion

Let $f(x)$ be infinitely differentiable at x_0. Then we can write its Taylor expansion:

$$f(x) \sim f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{55}$$

Now the question is, what is the relation between $f(x)$ and the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n$?

Lemma 14. *If $f(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ for all $x \in (x_0 - R, x_0 + R)$ for some $R > 0$, then*

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{56}$$

Proof. From the assumption we know that the radius of convergence $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ is at least R. Now according to the differentiability of power series,

$$f^{(k)}(x_0) = \sum_{n=0}^{\infty} \frac{\mathrm{d}^k[a_n\,(x - x_0)^n]}{\mathrm{d}x^k}|_{x = x_0} = (k!)\,a_k \tag{57}$$

and the conclusion follows. □

Corollary 15. *If there are $a < b$ such that $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n = 0$ for all $x \in (a, b)$, then $\forall n \in \mathbb{N}$, $a_n = 0$.*

Exercise 15. Prove: If $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n = \sum_{n=0}^{\infty} b_n\,(x - x_0)^n$ for all $x \in (a, b)$ where $x_0 \in (a, b)$, then $a_n = b_n$ for all n. (Hint:[11])

Thus if $f(x)$ equals a power series, then this power series must be its Taylor expansion. However the converse claim does not hold.

Example 16. Let

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{58}$$

Then its Taylor expansion is $\sum_{n=0}^{\infty} 0 \cdot x^n$ which obviously sum up to $0 \neq f(x)$ for all $x \neq 0$.

Proof. By definition of derivative we have

$$f'(0) = \lim_{x \to 0} \frac{\exp(-1/x^2) - 0}{x} = \lim_{t \to \infty} t\,e^{-t^2} = 0 \tag{59}$$

thanks to L'Hospitale.

On the other hand we have

$$f'(x) = \frac{2}{x^3} \exp(-1/x^2), \qquad x \neq 0. \tag{60}$$

11. Setting $x = x_0$ we have $a_0 = b_0$. Take derivative and then set $x = x_0$ we have $a_1 = b_1$. Prove by induction.

This gives

$$f''(0) = \lim_{x \to 0} \frac{\frac{2}{x^3} \exp(-1/x^2) - 0}{x} = \lim_{t \to \infty} 2\, t^4 \exp(-t^2) = 0 \tag{61}$$

again thanks to L'Hospitale.

In general, we can prove by induction that

$$f^{(k)}(x) = \begin{cases} P(1/x) \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{62}$$

where P is a polynomial. □

Example 17. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{n!}. \tag{63}$$

It is easy to prove that $f(x)$ is infinitely differentiable. However its Taylor expansion has radius of convergence 0.

Exercise 16. Prove that $f(x)$ is infinitely differentiable. (Hint:[12])

Exercise 17. Prove that its Taylor expansion has radius of convergence 0. (Hint:[13])

3.2. A sufficient condition

THEOREM 18. *Let $x_0 \in \mathbb{R}$ and $R > 0$. Let $f(x): (x_0 - R, x_0 + R) \mapsto \mathbb{R}$. Assume*

$$\exists M > 0, \qquad \forall n \in \mathbb{N}, \forall x \in (x_0 - R, x_0 + R) \qquad \left| f^{(n)}(x) \right| < M^n, \tag{66}$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tag{67}$$

in $(x_0 - R, x_0 + R)$. Note that here R could be ∞.

Proof. Left as exercise. □

Exercise 18. Prove the theorem. (Hint:[14])

Exercise 19. Can you obtain a similar theorem with weaker assumption? Justify.

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12. Prove $\lim_{n \to \infty} \frac{m}{(n!)^{1/n}} = 0$ for any fixed m. To do this, divide $n! = [n\ (n-1) \cdots (2\ m)][(2\ m-1) \cdots 1]$. Thus $\frac{m}{(n!)^{1/n}} < \frac{m}{(2m)^{(n-2m)/n}}$.

13. Direct calculation gives

$$f^{(2k+1)}(0) = \sum_{n=0}^{\infty} \frac{(2^{2k+1})^n}{n!} = \exp(2^{2k+1}). \tag{64}$$

Then prove

$$\lim_{n \to \infty} \left[\frac{\exp(2^n)}{n!} \right]^{1/n} \geqslant \lim_{n \to \infty} \frac{\exp(2^n/n)}{n} = \infty \tag{65}$$

14. Apply Taylor expansion with Lagrange type of remainder. Note that (67) is not a consequence of the radius of convergence being infinity! See Example 16.

Example 19. The following holds for all $x \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n; \tag{68}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}; \tag{69}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

Example 20. Find the Taylor expansion of $\arctan x$ at $x = 0$.

We notice that

$$(\arctan x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots \qquad |x| < 1. \tag{70}$$

Integrate termwise, we have

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{71}$$

For now this holds for $|x| < 1$.

Remark 21. (71) still holds at $|x| = 1$. This is a consequence of Theorem 24 (Abel's Theorem).

3.3. Sum and product of power series

THEOREM 22. *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n, \sum_{n=0}^{\infty} b_n (x - x_0)^n$ be power series with radius of convergence R_1, R_2. Denote on $(x_0 - R_1, x_0 + R_1)$ and $(x_0 - R_2, x_0 + R_2)$, respectively,*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n; \qquad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n. \tag{72}$$

Let $R = \min\{R_1, R_2\}$. Then for any $x \in (x_0 - R, x_0 + R)$, we have

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n; \qquad f(x)\, g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{73}$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

Exercise 20. In the above theorem we choose the index n to start from 0 to make the presentation simple. Explain why this is so.

Proof. Left as exercise. □

Exercise 21. Prove the above theorem.

Example 23. We have

$$\frac{1}{(1-x)(2-x)} = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) x^n, \qquad x \in (-1, 1); \tag{74}$$

$$\frac{\ln(1-x)}{1-x} = -\sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n, \qquad x \in (-1, 1). \tag{75}$$

The proofs are left as exercises.

Exercise 22. Justify every step in the solution of Airy's equation in Introduction.

4. FINER PROPERTIES AND TOPICS

4.1. Abel's Theorem

THEOREM 24. (ABEL) *Consider power series* $\sum_{n=1}^{\infty} a_n\,(x-x_0)^n$ *with radius of convergence* $R\in(0,\infty)$. *Denote its sum by* $S(x)$.

- *If* $\sum_{n=1}^{\infty} a_n\,R^n$ *converges, then*

$$\lim_{x\to(x_0+R)-} S(x) = S(x_0+R); \tag{76}$$

- *If* $\sum_{n=1}^{\infty} a_n\,(-R)^n$ *converges, then*

$$\lim_{x\to(x_0-R)+} S(x) = S(x_0-R). \tag{77}$$

Proof. We only prove the first claim and leave the second as exercise.

We prove that in this case the convergence on $[x_0, x_0+R]$ is in fact uniform. Write

$$\sum_{n=1}^{\infty} a_n\,(x-x_0)^n = \sum_{n=1}^{\infty} a_n\,R^n \left(\frac{x-x_0}{R}\right)^n. \tag{78}$$

The conclusion then follows from Abel's test for uniform convergence. □

Exercise 23. Prove the other case.

Example 25. Calculate

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots; \qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \tag{79}$$

Solution.

For the first one we notice

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = \ln(1+x). \tag{80}$$

By Abel's theorem we have

$$\sum_{n=1}^{\infty} \lim_{x\to 1}\left[(-1)^{n-1}\,\frac{x^n}{n}\right] = \lim_{x\to 1}\,[\ln(1+x)] = \ln 2. \tag{81}$$

For the second one we notice

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}\,x^{2n+1} \tag{82}$$

and therefore the sum is $\pi/4$.

4.2. Generating functions

The theory of power series form the foundation of the method of "generating functions" which is very useful in combinatorics, difference equations, probability theory, etc..

DEFINITION 26. (GENERATING FUNCTION) *Let* $\{a_n\}$ *be a sequence. Call the power series*

$$\sum_{n=0}^{\infty} a_n\,x^n = a_0 + a_1\,x + \cdots \tag{83}$$

its generating function.

Example 27. Prove

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{84}$$

Proof. Notice that the generating function for $\binom{n}{0}$, ..., $\binom{n}{n}$ is $(1+x)^n$. Now $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ is the coefficient of the x^n term in $(1+x)^n \cdot (1+x)^n$. Thus ends the proof. □

Exercise 24. Let $p, q, n \in \mathbb{N}$. Prove

$$\sum_{k=0}^{n} \binom{p+k}{p}\binom{q+n-k}{q} = \binom{p+q+n+1}{p+q+1}. \tag{85}$$

(Hint: [15])

Example 28. Find the general formula for a_n defined through

$$a_0 = 0, a_1 = 1, a_2 = -1, \qquad a_n = -a_{n-1} + 16\, a_{n-2} - 20\, a_{n-3}. \tag{86}$$

Solution. Let $f(x)$ be its generating function. Then we have

$$f(x) = x - x^2 + \cdots \tag{87}$$

and satisfies

$$(1 + x - 16\, x^2 + 20\, x^3)\, f(x) = x \Longrightarrow f(x) = \frac{x}{1 + x - 16\, x^2 + 20\, x^3}. \tag{88}$$

The general formula can then be obtained through partial fraction.

Exercise 25. Finish the calculation for the above example. (Answer: [16])

4.3. Uniform approximation

In 1885, Weierstrass proved the following.

THEOREM 29. *Let $f: [a, b] \mapsto \infty$ be continuous. Then for every $\varepsilon > 0$, there is a polynomial $P(x)$ such that*

$$\forall x \in [a, b], \qquad |f(x) - P(x)| < \varepsilon. \tag{90}$$

Exercise 26. Prove that, if for every $\varepsilon > 0$, there is a polynomial $P(x)$ such that

$$\forall x \in [a, b], \qquad |f(x) - P(x)| < \varepsilon, \tag{91}$$

then $f(x)$ is continuous on $[a, b]$.

Proof. We will prove the case $[a, b] = [0, 1]$ and furthermore $f(0) = f(1) = 0$, and leave the general case as exercise.

15. Consider $(1-x)^{-(p+1)}, (1-x)^{-(q+1)}$.

16. $$a_n = \frac{1}{7}(n+1)\, 2^n - \frac{1}{49}\, 2^{n+1} + \frac{(-1)^{n+1}}{49}\, 5^{n+1}. \tag{89}$$

Consider $Q_n(x) := \left[\int_{-1}^{1} (1-x^2)^n \,\mathrm{d}x\right]^{-1} (1-x^2)^n$. Then Q_n is a polynomial and satisfies

$$\int_{-1}^{1} Q_n(x)\,\mathrm{d}x = 1. \tag{92}$$

On the other hand, it is easy to prove that

$$\int_{-1}^{1} (1-x^2)^n \,\mathrm{d}x \geqslant 2\int_{0}^{1/\sqrt{n}} (1-n\,x^2)\,\mathrm{d}x > \frac{1}{\sqrt{n}} \Longrightarrow c_n < \sqrt{n}. \tag{93}$$

Now define

$$\tilde{f}(x) := \begin{cases} f(x) & x \in [0,1] \\ 0 & \text{elsewhere} \end{cases} \tag{94}$$

and

$$P_n(x) = \int_{-1}^{1} \tilde{f}(x+t)\,Q_n(t)\,\mathrm{d}t. \tag{95}$$

It can be shown that $P_n(x)$ is a polynomial. Now we have

$$\left|P_n(x) - \tilde{f}(x)\right| = \left|\int_{-1}^{1} \left[\tilde{f}(x+t) - \tilde{f}(x)\right] Q_n(t)\right| \leqslant \int_{-1}^{1} \left|\tilde{f}(x+t) - \tilde{f}(x)\right| Q_n(t)\,\mathrm{d}t. \tag{96}$$

For any $\varepsilon > 0$, since $\tilde{f}(x)$ is uniformly continuous, there is $\delta > 0$ such that

$$\forall |t| < \delta, \qquad \left|\tilde{f}(x+t) - \tilde{f}(x)\right| < \frac{\varepsilon}{2}. \tag{97}$$

On the other hand, we have

$$\forall |x| \in [\delta, 1], \qquad Q_n(x) < \sqrt{n}\,(1-\delta^2)^n. \tag{98}$$

Thus

$$\int_{-1}^{1} \left|\tilde{f}(x+t) - \tilde{f}(x)\right| Q_n(t)\,\mathrm{d}t < \frac{\varepsilon}{2} + 4\,M\,\sqrt{n}\,(1-\delta^2)^n \tag{99}$$

where $M := \max_{x\in[0,1]} |f(x)|$ and the conclusion follows. □

Exercise 27. Prove the general case.

Exercise 28. Prove that $\tilde{f}(x)$ is uniformly continuous.

Exercise 29. Prove that $P_n(x)$ is a polynomial.

Exercise 30. Fill in the details for the last step.

Remark 30. For those who knows the Dirac δ-function and convolution, please notice that $Q_n(x)$ approximates the Dirac δ function, and P_n is defined as the convolution of $\tilde{f}(x)$ and $Q_n(x)$.

Exercise 31. (For those to whom the above remark makes sense) Prove the theorem using a different sequence of $Q_n(x)$.

5. MORE EXERCISES AND PROBLEMS

5.1. Basic exercises

Exercise 32. (USTC3) Find the radius of convergence for the following power series, and study their convergence/divergence at the ends points. In the following $a, b > 0$.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2\,n)} x^n; \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n; \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) x^n \tag{100}$$

$$\sum_{n=1}^{\infty} \left(\frac{a^n}{n}+\frac{b^n}{n^2}\right) x^n; \qquad \sum_{n=1}^{\infty} \frac{x^n}{a^n+b^n}; \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{n}{e}\right)^n x^n. \tag{101}$$

(For the last one you need Stirling's formula) (Ans:[17])

Exercise 33. (USTC3) Let R_1, R_2 be the radii of convergence for $\sum_{n=0}^{\infty} a_n\, x^n$ and $\sum_{n=0}^{\infty} b_n\, x^n$, respectively. Prove

a) The radius of convergence R_3 for $\sum_{n=0}^{\infty} (a_n + b_n)\, x^n$ satisfies $R_3 \geqslant \min(R_1, R_2)$;

b) The radius of convergence R_4 for $\sum_{n=0}^{\infty} (a_n\, b_n)\, x^n$ satisfies $R_4 \geqslant R_1\, R_2$;

c) Show through examples that in both cases $>$ may hold.

Exercise 34. (USTC3) Calculate

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2\,n+1}; \qquad \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{2\,n+1}; \qquad \sum_{n=1}^{\infty} \frac{x^n}{n\,(n+1)}. \tag{102}$$

Justify your calculation. (Hint:[18])

Exercise 35. (USTC3) Prove the following for $x \in (-1, 1)$.

$$\sum_{n=0}^{\infty} (n+1)\,(n+2)\, x^n = \frac{2}{(1-x)^3}; \qquad \sum_{n=1}^{\infty} n^3\, x^n = \frac{x+4\,x^2+x^3}{(1-x)^4}. \tag{103}$$

Exercise 36. Let $k \in \mathbb{N} \cup \{0\}$. The Bessel function of order k is defined as

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+k)!} \left(\frac{x}{2}\right)^{2n+k}. \tag{104}$$

a) Calculate the radius of convergence of the series on the RHS;

b) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^k\, J_k(x)] = x^k\, J_{k-1}(x); \qquad \frac{\mathrm{d}}{\mathrm{d}x}[x^{-k}\, J_k(x)] = -x^{-k}\, J_{k+1}(x). \tag{105}$$

Justify your answers;

c) Prove that $J_k(x)$ solves the equation

$$x^2\, y'' + x\, y' + (x^2 - k^2)\, y = 0. \tag{106}$$

Note that a "proof" involves more than formally substituting the series into the equation and do termwise differentiation.

5.2. More exercises

Exercise 37. Let $\alpha \in \mathbb{R}$. Through studying the expansion of $(1+x)^\alpha$, prove the following generalized binomial expansion:

$$(x+y)^\alpha = x^\alpha + \sum_{n=1}^{\infty} \frac{\alpha\,(\alpha-1)\cdots(\alpha-n+1)}{n!}\, y^n\, x^{\alpha-n}. \tag{107}$$

17. $0, e^{-1}, 1, [\max\{|a|, |b|\}]^{-1}, \max\{|a|, |b|\}, 1$.

18. Termwise differentiation for the first two; For the last one multiply by x and then differentiate twice.

Exercise 38. Show that Abel's Theorem does not hold for general infinite series of functions.

Exercise 39. (USTC3) Prove the following:

$$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\ln 2\right);\quad 1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\cdots=\frac{1}{4\sqrt{2}}\left(\pi+2\ln\left(\sqrt{2}+1\right)\right). \tag{108}$$

Exercise 40. (USTC3) Use $\int_0^1 t^{n-1}(1-t)\,\mathrm{d}t=\frac{1}{n\,(n+1)}$ to prove

$$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots=\frac{\pi}{4}+\frac{1}{2}\ln 2. \tag{109}$$

Exercise 41. Show that there is no f satisfying both of the following:

- There is a sequence of numbers $\{a_n\}$ such that

$$f(x)=\sum_{n=0}^{\infty} a_n\,x^n \qquad x\in(-1,1); \tag{110}$$

- $f\left(\frac{1}{n}\right)=\frac{(-1)^n}{n}$.

(Hint:[19])

Exercise 42. Show that there is no f satisfying both of the following:

- There is a sequence of numbers $\{a_n\}$ such that

$$f(x)=\sum_{n=0}^{\infty} a_n\,x^n \qquad x\in(-1,1); \tag{111}$$

- $f\left(\frac{1}{n}\right)=\frac{\sin n}{n^2}$.

What if we replace $\sin n$ by $\sin\left(\frac{1}{n}\right)\sin n$?

5.3. Problems

Problem 1. (USTC3) Determine the points of convergence for the following infinite series:

$$\sum_{n=0}^{\infty}\frac{1}{2\,n+1}\left(\frac{1-x}{1+x}\right)^n;\qquad \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}e^{-n\,x};\qquad \sum_{n=1}^{\infty}\left(\frac{1}{x}\right)^n\sin\left(\frac{\pi}{2^n}\right). \tag{112}$$

Justify your answers.

Problem 2. (USTC3) Let $a_n\geqslant 0$ for all n. Let the radius of convergence of $\sum_{n=0}^{\infty} a_n\,x^n$ be R. Prove

$$\lim_{x\to R-}\sum_{n=0}^{\infty} a_n\,x^n=\sum_{n=0}^{\infty} a_n\,R^n. \tag{113}$$

Note that R could be ∞ and the limit could also be infinity. Use this to prove $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$.

Problem 3. (USTC3) Let $c_n:=a_0\,b_n+a_1\,b_{n-1}+\cdots+a_n\,b_0$. Assume all three infinite series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$ are convergent. Then

$$\sum_{n=0}^{\infty} c_n=\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right). \tag{114}$$

Problem 4. (USTC3) Let f be infinitely differentiable on $[0,a]$ for some $a>0$. Further assume f and all its derivatives are non-negative on $[0,a]$. Prove that

$$\forall x\in[0,a),\qquad f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\,x^n. \tag{115}$$

Problem 5. (USTC3) Let $f\colon[a,b]\mapsto\mathbb{R}$ be continuous. Assume

$$\forall n\in\mathbb{N}\cup\{0\},\qquad \int_a^b x^n\,f(x)\,\mathrm{d}x=0, \tag{116}$$

19. Use intermediate value theorem to show the existence of $x_n\to 0$ such that $f(x_0)=0$. Then prove all $a_n=0$. Note that since f equals a power series on $(-1,1)$ it is necessarily continuous and therefore $f(0)=0$.

then $f(x)=0$ on $[a,b]$. (Hint:[20])

Problem 6. (USTC3) Let $f:\mathbb{R}\mapsto\mathbb{R}$. If for every $\varepsilon>0$, there is a polynomial $P(x)$ such that

$$\forall x\in\mathbb{R}, \qquad |f(x)-P(x)|<\varepsilon, \tag{117}$$

then $f(x)$ is a polynomial.

Problem 7. (EULER) In his 1734 paper *De progressionibus harmonicis observationes* (Observations on harmonic progressions)[21], Leonhard Euler did the following manipulations on harmonic-type series. Comment on his methods and results. If the results are right while the methods are not, give correct proofs of the results.

He considered "not simple harmonic progressions but combined with geometric ones"

$$s=\frac{c}{a}x+\frac{c}{a+b}x^2+\cdots \tag{118}$$

Multiply by $b\,x^{(a-b)/b}$ and differentiates, sum the resulting geometric series and then integrate:

$$s=\frac{c}{b\,x^{(a-b)/b}}\int\frac{x^{(a-b)/b}}{1-x}\,\mathrm{d}x. \tag{119}$$

Similarly, if

$$t=\frac{f}{g}x^m+\frac{f}{g+h}x^{2m}+\cdots \tag{120}$$

Euler obtained

$$t=\frac{f\,m}{h\,x^{m(g-h)/h}}\int\frac{x^{(mg-h)/h}}{1-x^m}\,\mathrm{d}x. \tag{121}$$

Taking $a=b=c=f=g=h=1$, he obtained

$$s-t=\ln\left(\frac{1-x^m}{1-x}\right)=\ln(1+x+\cdots+x^{m-1}). \tag{122}$$

Taking $m=2$ he obtained

$$\ln(1+x)=x-\frac{x^2}{2}+\cdots. \tag{123}$$

Setting $x=1$ he obtained

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\cdots \tag{124}$$

Setting $m=3$ and then $x=1$ he obtained a similar formula for $\ln 3$.

Euler set $h=2, a=b=g=c=1$, $f=2$ and $m=n$ to obtain

$$\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)-\left(\frac{2\,x^n}{1}+\frac{2\,x^{2n}}{3}+\cdots\right)=\ln\left(\frac{1-x^n}{(1-x)\,(1+x^n)}\right). \tag{125}$$

Taking $n=2$ and $x=0$ he reached

$$0=1-\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{3}{6}+\cdots. \tag{126}$$

20. Prove $\int_a^b f(x)^2\,\mathrm{d}x=0$.

21. Comm. Acad. Sci. Imp. Petropol. 7 (1734/5) 1740, 150-161.