Math 317 Week 01 Real Infinite Series

Last Update: February 27, 2014

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Main references

- (BINMORE) Binmore, K. G., *Mathematical Analysis: A Straightforward Approach*, Cambridge University Press, 1977, Chapter 6.
- (FOLLAND) Folland, Gerald B., *Advanced Calculus*, Prentice-Hall, Inc., 2002. Chapter 6.
- (KLINE) Kline, Morris, *Mathematical Thoughts from Ancient to Modern Times*, Oxford University Press, 1972. Chapter 20.
- (RIS) Bonar, Daniel D., Khoury, Michael J., *Real Infinite Series*, The Mathematical Association of America, 2006.
- (USTC3) He, Chen, Shi, Jihuai, Xu, Senlin, *ShuXueFenXi* (Mathematical Analysis) III, Higher Education Press, 1985, Section 8.1.
- (PKU2) Shen, Xiechang, *ShuXueFenXi* (Mathematical Analysis) II, Higher Education Press, 1986, Chapter 11.
- (BABY RUDIN) Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill Companies, Inc., 3rd edition, 1976. Chapter 3.
- (EULER) Sandifer, C. Edward, *The Early Mathematics of Leonhard Euler*, MAA, 2007.

1. Definitions and properties

1.1. Infinite series (Infinite sum)

Definition 1. (Infinite series) *Given a sequence $\{a_n\}$ of real numbers, the formal sum*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

is called an "infinite series".

Remark 2. Note that $\sum_{n=1}^{\infty} a_n$ is **just another way** of writing the formal sum $a_1 + a_2 + \cdots + a_n + \cdots$. They mean exactly the same thing (which, up to now, is – nothing).

Remark 3. The "summation" in (1) of these infinitely many real numbers

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{2}$$

is "formal" because it is not clear what it means to say $a_1 + a_2 + \cdots + a_n + \cdots = s \in \mathbb{R}$. Essentially, to define the "sum" of $a_1, \ldots, a_n, \ldots$ is to define a function

$$f: \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \cdots \mapsto \mathbb{R}. \tag{3}$$

This turns out to be a tricky task, as can be seen from the following example.

Example 4. Some examples of infinite series:

$$\sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + 1 + \cdots \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{5}$$

$$\sum_{n=1}^{\infty} 2^n = 1 + 2 + 4 + \cdots \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \tag{7}$$

It is intuitively clear that the value of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ should be 2 while $\sum_{n=1}^{\infty} 2^n$ should be ∞. It's not clear whether the first two sums correspond to any value. In particular, many different values can be "reasonably"[1] assigned to the first sum.

Example 5. (RIS, §6.3, Fallacy 7) Let $\sum_{n=1}^{\infty} a_n$ be any infinite series. Let $S \in \mathbb{R}$ be arbitrary. We have

$$a_1 = S - (S - a_1); \quad a_2 = (S - a_1) - (S - a_1 - a_2); \ldots \tag{8}$$

Everything cancel in the sum except S. Therefore $S = \sum_{n=1}^{\infty} a_n$.

Remark 6. For more such fallacious results, check out Chapter 6 of (RIS) .

1. Meaning: following accepted rules of adding finitely many numbers, such as grouping and re-arrangement.

1.2. Convergence through partial sum

One good way of defining infinite sum is through the limit of partial sums.

DEFINITION 7. (PARTIAL SUM AND CONVERGENCE) *The nth partial sum of an infinite series* $\sum_{n=1}^{\infty} a_n$ *is defined as* $s_n = \sum_{m=1}^{n} a_m$. *If the sequence* $\{s_n\}$ *converges to some real number* s, *then we say the infinite series converges to* s, *or equivalently say its sum is* s, *and simply write*

$$\sum_{n=1}^{\infty} a_n = s. \tag{9}$$

If $s \longrightarrow \infty$ *or* $-\infty$, *we say the infinite series diverges to* ∞ *or* $-\infty$ *respectively and write*

$$\sum_{n=1}^{\infty} a_n = \infty \text{ or } -\infty. \tag{10}$$

Remark 8. Note that

$$s_1 = a_1, s_2 = a_1 + a_2, s_3 = a_1 + a_2 + a_3, \ldots \tag{11}$$

Recalling theorems for the convergence of sequences, we have

THEOREM 9.

- $\sum_{n=1}^{\infty} a_n = s$ *if and only if for any* $\varepsilon > 0$, *there is* $N \in \mathbb{N}$ *such that for all* $n > N$,

$$\left| s - \sum_{m=1}^{n} a_m \right| < \varepsilon; \tag{12}$$

- $\sum_{n=1}^{\infty} a_n = \infty$ *if and only if for any* $M \in \mathbb{R}$, *there is* $N \in \mathbb{N}$ *such that for all* $n > N$,

$$\sum_{m=1}^{n} a_m > M; \tag{13}$$

- $\sum_{n=1}^{\infty} a_n = -\infty$ *if and only if for any* $M \in \mathbb{R}$, *there is* $N \in \mathbb{N}$ *such that for all* $n > N$,

$$\sum_{m=1}^{n} a_m < M. \tag{14}$$

Example 10. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \tag{15}$$

Proof. By induction we can prove that

$$s_n = 1 - \frac{1}{2^n} \tag{16}$$

and the conclusion easily follows. □

Example 11. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$ converges and find the value.

Proof. By induction we can prove that

$$s_n = 1 - \frac{1}{n+1} \tag{17}$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$. □

Exercise 1. Study the convergence/divergence of $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$. (Hint: [2]).

Exercise 2. Prove the following:

i. A convergent series has a unique sum;

ii. If $\sum_{n=1}^{\infty} a_n = A \in \mathbb{R} \cup \{\pm\infty\}$, and $c \in \mathbb{R}$, $c \neq 0$, then $\sum_{n=1}^{\infty} (c\, a_n) = c\, A$;

iii. If $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$, then $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$;

1.3. Criteria for convergence

THEOREM 12. (CAUCHY) *A infinite series $\sum_{n=1}^{\infty} a_n$ converges to some $s \in \mathbb{R}$ if and only if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m > n > N$,*

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{18}$$

Exercise 3. Prove the above theorem. (Hint:[3])

COROLLARY 13. *If $\sum_{n=1}^{\infty} a_n$ converges to $s \in \mathbb{R}$ then $\lim_{n \longrightarrow \infty} a_n = 0$. Equivalently, if $\lim_{n \to \infty} a_n$ does not exist, or exists but is not 0, then $\sum_{n=1}^{\infty} a_n$ does not converge to any real number.*

Proof. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} a_n$ converges, it is Cauchy and there exists $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{19}$$

Now take $N = N_1 + 1$. Then for any $n > N$, we have $n > n - 1 > N_1$ which gives

$$|a_n - 0| = \left| \sum_{k=n}^{n} a_k \right| < \varepsilon. \tag{20}$$

Thus by definition of convergence of sequence $\lim_{n \longrightarrow \infty} a_n = 0$. □

Remark 14. The above corollary is very useful, however we should keep in mind that:

1. The converse is not true. That is $\lim_{n \to \infty} a_n = 0$ does not imply the convergence of $\sum_{n=1}^{\infty} a_n$.
2. It cannot be applied to conclude $\sum_{n=1}^{\infty} a_n \neq \infty$ or $-\infty$.

Exercise 4. Give a counterexample to the following claims:

$$\lim_{n \longrightarrow \infty} a_n = 0 \Longrightarrow \sum_{n=1}^{\infty} a_n = s \text{ for some } s \in \mathbb{R}. \tag{21}$$

$$\sum_{n=1}^{\infty} a_n = s \in \mathbb{R}_{\text{ext}} \Longrightarrow \lim_{n \longrightarrow \infty} a_n = 0. \tag{22}$$

2. $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1} - \sqrt{n}$

3. Recall Definition 7. Then check the Cauchy criterion for infinite sequence.

(Hint:[4])

Example 15. Let $a_n = c\, r^{n-1}$ for $r, c \in \mathbb{R}$. Then

a) If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = \frac{c}{1-r}$.

b) If $r \geqslant 1$, then $\sum_{n=1}^{\infty} a_n = \begin{cases} +\infty & c > 0 \\ 0 & c = 0 \\ -\infty & c < 0 \end{cases}$.

c) If $r \leqslant -1$, then $\sum_{n=1}^{\infty} a_n$ does not exist (as extended real number).

Proof. We prove for $c = 1$. The generalization is trivial.

a) We have

$$\sum_{m=1}^{n} a_m = 1 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}. \tag{23}$$

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N \geqslant \log_{|r|} [\varepsilon (1 - r)]$, then for any $n > N$,

$$\left| \frac{1}{1-r} - \sum_{m=1}^{n} a_m \right| = \frac{|r|^n}{1-r} < \frac{|r|^N}{1-r} < \varepsilon. \tag{24}$$

b) For any $M \in \mathbb{R}$. Take $N \in \mathbb{N}$ such that $N > |M|$. Then for every $n > N$ we have

$$\sum_{m=1}^{n} a_m \geqslant \sum_{m=1}^{n} 1 = n > N > |M| \geqslant M. \tag{25}$$

c) Since $r \leqslant -1$, $|a_n| \geqslant 1$. Therefore by Corollary 13 $\sum_{n=1}^{\infty} a_n$ does not converge to any real number. We still need to show that $\sum_{n=1}^{\infty} a_n \neq \infty, -\infty$. To do this, we show that $s_n = \sum_{m=1}^{n} a_n$ satisfies $s_n \geqslant 0$ when n is odd and $s_n \leqslant 0$ when n is even. Clearly $s_1 = 1 > 0$, $s_2 = 1 + r \leqslant 0$. For $n \geqslant 3$, calculate

$$s_n = \sum_{m=1}^{n-1} r^{m-1} + r^n = \frac{1 - r^{n-1}}{1 - r} + r^n. \tag{26}$$

As $r \leqslant -1$, $1 - r \geqslant 2$ which gives

$$\left| \frac{1 - r^{n-1}}{1 - r} \right| \leqslant \frac{1 + |r|^{n-1}}{2} \leqslant |r|^{n-1} \leqslant |r|^n. \tag{27}$$

Therefore, s_n and r^n cannot take opposite signs. $\square$

Example 16. We have

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2; \qquad \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{5}{4}. \tag{28}$$

4. $\sum \frac{1}{n}$; $\sum 1$.

2. NON-NEGATIVE SERIES

2.1. Convergence/divergence through comparison

In most situations, it is impossible – at least very hard – to explicitly calculate the partial sum $S_n := \sum_{m=1}^{n} a_m$ and it is therefore not possible to establish convergence/find the sum based on definition only. One way to overcome this is through the following comparision theorem.

THEOREM 17. (COMPARISON) *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series. Assume that there are $c > 0$ and $N_0 \in \mathbb{N}$ such that $|a_n| \leqslant c\, b_n$ for all $n > N_0$. Then*

a) $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

b) $\sum_{n=1}^{\infty} a_n$ does not converge[5] $\Longrightarrow \sum_{n=1}^{\infty} b_n = \infty$.

Proof. Note that a) and b) are equivalent logical statements[6], so we only need to prove a). We show that $\sum_{n=1}^{\infty} a_n$ is Cauchy. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} b_n$ converges, it is Cauchy and there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} b_k \right| < \frac{\varepsilon}{c}. \tag{29}$$

Take $N = \max\{N_1, N_0\}$. Then for any $m > n > N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| \leqslant \sum_{k=n+1}^{m} |a_k| \leqslant c \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon. \tag{30}$$

So $\sum_{n=1}^{\infty} a_n$ is Cauchy and therefore converges. □

Exercise 5. Show that the $\Longrightarrow$ in the above cannot be replaced by $\Longleftrightarrow$. Also show that the absolute value is necessary in $|a_n| \leqslant c\, b_n$. (Hint: [7])

Exercise 6. Let $\sum_{n=1}^{\infty} a_n$ be a non-negative series. Then it converges $\Longleftrightarrow$ it is bounded above. (Hint:[8])

Example 18. It is clear by the above theorem that if $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$. The converse is not true, as can be seen from the following example:

Take $a_n = \frac{(-1)^{n+1}}{n}$. Then we clearly see that

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2\,n-1} - \frac{1}{2\,n} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2\,n-1)\,(2\,n)} \tag{31}$$

5. Note that here it is not necessary for $\sum a_n$ to be ∞.

6. By the hypotheses in the Theorem, $b_n \geqslant 0$. See Exercise 6.

7. Consider $a_n = -1/n^2$, $b_n = 1/n$.

8. Thus the partial sum $\sum_{m=1}^{n} a_m$ is increasing and there are only two cases: bounded above or not.

converges. On the other hand, we have $S_{2n+1} - S_{2n} \longrightarrow 0$ so S_{2n+1} converges to the same limit. From here it is easy to prove by definition that $S_n \longrightarrow$ to the same limit, which turns out to be $\ln 2$.

Example 19. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty. \tag{32}$$

Proof. We notice that

$$\frac{1}{\sqrt{n}} > \frac{2}{\sqrt{n} + \sqrt{n+1}}. \tag{33}$$

The conclusion follows from $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} + \sqrt{n+1}} = \infty$. □

Exercise 7. Prove the following "limit comparison test".

Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be two non-negative series and further assume $b_n > 0$. Assume that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \in \mathbb{R} \cup \{+\infty\}. \tag{34}$$

Then

i. If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ both converge or both diverge;

ii. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges;

iii. If $L = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Then improve the result using limsup and liminf. (Hint:[9])

Exercise 8. Apply the limit comparison test to study $\sum_{n=1}^{\infty} \frac{2^n + n - 2}{3^n + 4n - 5}$.

Exercise 9. Can we drop the "non-negative" assumption in the above "limit comparison test"? (Hint:[10])

Remark 20. A sequence $\sum_{n=1}^{\infty} a_n$ that converges but with $\sum_{n=1}^{\infty} |a_n| = \infty$ is called c*onditionally convergent*. On the other hand, a sequence $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} |a_n|$ converges is said to be *absolutely convergent*. Absolutely convergent sequences can undergo any re-arrangement and still converge to the same sum but conditionally convergent sequences do not enjoy such property.

Exercise 10. Prove that if $\sum_{n=1}^{\infty} a_n$ does not converge to some $s \in \mathbb{R}$, then $\sum_{n=1}^{\infty} |a_n| = \infty$.

Remark 21. (The plan) In light of the Theorem 17, it is important to study *non-negative* series, that is infinite series $\sum_{n=1}^{\infty} a_n$ satisfying $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Once a non-negative sequence $\sum_{n=1}^{\infty} b_n$ is shown to be convergent, we know that any $\sum_{n=1}^{\infty} a_n$ satisfying $|a_n| \leqslant c\, b_n$ for some constant c is also convergent. It is further possible to make this comparison "intrinsic", that is design some criterion involving a_n only while still guarantees the relation $|a_n| \leqslant c\, b_n$. Such criteria are usually called "tests". We will study the simpliest tests in the following section.

9. For example, when $0 < L < \infty$, there is $N \in \mathbb{N}$ such that for all $n > N$, $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$.

10. No. Consider $a_n = 1/n, b_n = (-1)^n / \sqrt{n}$.

2.2. Typical non-negative series and their implications

Example 22. (GEOMETRIC SERIES) We have seen that $\sum_{n=1}^{\infty} r^{n-1}$ converges when $0 \leqslant r < 1$. As a consequence, if another series $\sum_{n=1}^{\infty} a_n$ satisfies

$$|a_n| \leqslant c\, r^{n-1} \tag{35}$$

for some $c > 0$ and for all $n >$ some $N_0 \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.

Example 23. (GENERALIZED HARMONIC SERIES/p-SERIES) The series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges when $a > 1$ and diverges when $a \leqslant 1$.

Proof. When $a \leqslant 1$, we have $\frac{1}{n^a} \geqslant \frac{1}{n}$ therefore it suffices to show the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$. We use the following trick:[11]

$$\begin{aligned} 1 &> \frac{1}{2}, & (36)\\ \frac{1}{2} + \frac{1}{3} &> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, & (37)\\ \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} &> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}, & (38)\\ \vdots \quad &\vdots \quad \vdots \end{aligned}$$

Therefore we have (recall s_n is the partial sum)

$$s_{n_k} > \frac{k+1}{2} \tag{39}$$

where $n_k = 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$. Therefore s_n is not bounded and the series diverges to ∞.

When $a > 1$, we use the following trick:

$$\begin{aligned} 1 &\leqslant 1; & (40)\\ 2^{-a} + 3^{-a} &< 2 \cdot 2^{-a} = 2^{1-a}; & (41)\\ 4^{-a} + 5^{-a} + 6^{-a} + 7^{-a} &< 4 \cdot 4^{-a} = 2^{2(1-a)}; & (42)\\ \vdots \quad &\vdots \quad \vdots \end{aligned}$$

We see that

$$\sum_{n=1}^{N} \frac{1}{n^a} < \sum_{n=1}^{\infty} \left(2^{(1-a)}\right)^{n-1} = \frac{1}{1 - 2^{1-a}} < +\infty. \tag{43}$$

11. This proof is attributed to Nicole Oresme (c1320-1325 – 1382), whose motivation may be "If God were Infinity, then divergent series would be His angels flying higher and higher to reach Him." The result was re-proved many times, including one in the book *Tractatus de Seriebus Infinitis (Treatise on Infinite Series)* by Jacob Bernoulli, who commented "So the soul of immensity dwells in minutia. And in narrowest limits no limits inhere. What joy to discern the minute in infinity! The vast to perceive in the small, what divinity!" (Quotes from Clifford A. Pickover, The MαTH βOOK, p.104.)

According to (KLINE) the proof appeared in Oresme: *Questiones Super Geometriam Euclidis* (c. 1360).

Thus this non-negative series is bounded from above and therefore converges. □

Remark 24. Usually the convergence/divergence of generalized harmonic series is studied through the following relation between non-negative series and improper integrals.

THEOREM. (INTEGRAL TEST) *Consider a series* $\sum_{n=1}^{\infty} a_n$*. Assume there is a function* $f: [1, \infty) \mapsto \mathbb{R}^+$ *such that* $f(n) = a_n$*. Further assume* f *to be integrable and decreasing. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\,\mathrm{d}x \tag{44}$$

converges. Furthermore in the case of convergence, we have

$$\sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,\mathrm{d}x < \sum_{n=1}^{\infty} a_n. \tag{45}$$

We see here that it can be done directly. It is also worth mentioning that the technique in the above proof is a special case of the so-called Cauchy's Condensation Test.

Exercise 11. Prove the "Integral test" theorem. (Hint:[12])

Remark 25. The same method can show that

$$\sum_{n=1}^{\infty} \frac{1}{n \log(n+1)^a}, \quad \sum_{n=1}^{\infty} \frac{1}{n \log(n+1) \log[\log(n+1)+1]^a}, \ldots \tag{47}$$

converges when $a > 1$ and diverges when $a \leqslant 1$.

Exercise 12. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n \log(n+1)^a} \tag{48}$$

is convergent for $a > 1$ and divergent for $a \leqslant 1$ using two methods: Directly and through comparing against certain improper integrals. (Hint:[13])

12. Notice

$$a_n \leqslant \int_n^{n+1} f(x)\,\mathrm{d}x \leqslant a_{n+1}. \tag{46}$$

13. For the direct proof, when $a > 1$, show that

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k \log(k+1)^a} \leqslant 2^n \left(\frac{1}{2^n n^a}\right) = \frac{1}{n^a}. \tag{49}$$

Thus this positive series is bounded above.

3. TESTS

3.1. Ratio and root tests

The following two intrinsic "convergence tests" based on comparison with geometric series are the simplest and most popular tests for convergence/divergence.

THEOREM 26. (RATIO TEST) *Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Further assume that $a_n \neq 0$ for all $n \in \mathbb{N}$. Then*

- *If $\limsup_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges.*
- *If $\liminf_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the series diverges.*

Proof.

- Assume $\limsup_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Set $r = \frac{L+1}{2}$ and $\varepsilon_0 = \frac{1-L}{2}$. By definition

$$\limsup_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \longrightarrow \infty} \left\{\sup_{k \geqslant n} \left|\frac{a_{k+1}}{a_k}\right|\right\} \tag{50}$$

therefore there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\sup_{k \geqslant n} \left|\frac{a_{k+1}}{a_k}\right| - L\right| < \varepsilon_0 \tag{51}$$

which gives

$$\left|\sup_{n > N} \left|\frac{a_{n+1}}{a_n}\right|\right| < L + \varepsilon_0 = r < 1 \Longrightarrow 0 < \left|\frac{a_{n+1}}{a_n}\right| < r < 1. \tag{52}$$

This gives, for all $n > N+1$,

$$|a_n| < |a_{N+1}|\, r^{n-N-1} = \frac{|a_{N+1}|}{r^N}\, r^{n-1}. \tag{53}$$

Note that since N is fixed, we have

$$|a_n| < c\, r^{n-1} \tag{54}$$

for all $n > N$ and consequently $\sum_{n=1}^{\infty} a_n$ converges.

- Assume $\liminf_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Set $\varepsilon_0 = L - 1$. Then by definition, similar to the limsup case above, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \tag{55}$$

which means for all $n \geqslant N+1$

$$|a_n| \geqslant |a_{N+1}| \tag{56}$$

As a consequence $a_n \not\longrightarrow 0$. By Corollary 13 we know that $\sum_{n=1}^{\infty} a_n$ diverges. □

Example 27. Prove that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ converges.

Proof. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\sqrt{n+1}} \tag{57}$$

therefore

$$\limsup_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1. \tag{58}$$

So the series converges. □

Exercise 13. Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Let $\sum_{n=1}^{\infty} b_n$ be the series obtained by dropping all 0's from $\{a_n\}$. Prove that

$$\sum_{n=1}^{\infty} a_n = s \Longleftrightarrow \sum_{n=1}^{\infty} b_n = s \tag{59}$$

for $s \in \mathbb{R}_{\text{ext}}$. (Hint:[14])

Now use this result to generalize the ratio test to all infinite series without assuming $a_n \neq 0$.

Note that both are limsup *! Root test is very special in this sense.*

THEOREM 28. (ROOT TEST) *Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Then*

- *If* $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} < 1$*, then the series converges.*
- *If* $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} > 1$*, then the series diverges.*

Proof.

- Assume $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} = L < 1$. Set $r = \frac{L+1}{2}$ and $\varepsilon_0 = \frac{1-L}{2}$. Then by definition, as in the proof of the above ratio test, there is $N \in \mathbb{N}$ such that for all $n > N$,

  $$|a_n|^{1/n} < r < 1 \Longrightarrow |a_n| < r^n. \tag{60}$$

  Therefore $\sum_{n=1}^{\infty} a_n$ converges.
- Assume $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} > 1$. The proof is left as exercise. □

Remark 29. It turns out that for any sequence $\{x_n\}$,

$$\liminf_{n\longrightarrow\infty}\left|\frac{x_{n+1}}{x_n}\right| \leqslant \liminf_{n\longrightarrow\infty} |x_n|^{1/n} \leqslant \limsup_{n\longrightarrow\infty} |x_n|^{1/n} \leqslant \limsup_{n\longrightarrow\infty}\left|\frac{x_{n+1}}{x_n}\right|. \tag{61}$$

Therefore the root test is sharper than the ratio test, in the sense that any series that passes the ratio test for convergence (or divergence) will also pass the root test.

Example 30. Consider the infinite series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} 2^{-k} & n = 2k-1 \\ 3^{-k} & n = 2k \end{cases}$. It can be easily verified that

$$\liminf_{n\longrightarrow\infty}\frac{a_{n+1}}{a_n} = 0; \qquad \limsup_{n\longrightarrow\infty}\frac{a_{n+1}}{a_n} = \infty \tag{62}$$

so the ratio test does not apply. On the other hand

$$\limsup_{n\longrightarrow\infty} (a_n)^{1/n} = \frac{1}{\sqrt{2}} \tag{63}$$

so the root test tells us that the series converges.

3.2. Raabe's test and Gauss's test

Note that neither the ratio test nor the root test works for the generalized harmonic series.

Exercise 14. Verify this claim. (Hint:[15])

14. The partial sums of $\sum b_n$ form a subsequence of the sequence of partial sums of $\sum a_n$.

THEOREM 31. (RAABE'S TEST) [16] *Let* $\sum_{n=1}^{\infty} a_n$ *be a positive infinite series. Then*

- $\sum_{n=1}^{\infty} a_n$ *converges if*
$$\liminf_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)>1 \tag{64}$$
- $\sum_{n=1}^{\infty} a_n$ *diverges if*
$$\limsup_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)<1. \tag{65}$$

Exercise 15. Consider
$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2\,n-1)}{2\cdot 4\cdot 6\cdots(2\,n)} \text{ and } \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2\,n-1)}{2\cdot 4\cdot 6\cdots(2\,n)\,(n+1)}. \tag{66}$$
Prove that neither the ratio test nor the root test applies. Apply Raabe's test to prove that the first one diverges but the second one converges.

Exercise 16. Let $p>0$. Prove that neither the ratio test nor the root test applies. Apply Raabe's test to prove that
$$\sum_{n=1}^{\infty} \frac{p\,(p+1)\cdots(p+n-1)}{n!} \tag{67}$$
diverges.

Exercise 17. Let $x>0$. Study the series
$$\sum_{n=1}^{\infty} \frac{n!}{(x+1)\cdots(x+n)}. \tag{68}$$
(Hint: [17])

Proof. We prove the convergence part and leave the other half as exercise.

Since $\liminf_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)>1$, there is $\beta>0$ such that
$$n\left(\frac{a_n}{a_{n+1}}-1\right)>1+\beta \tag{70}$$
for all $n>$ some $N_0\in\mathbb{N}$. Thus we have for such n,
$$\frac{a_n}{a_{n+1}}>1+\frac{1+\beta}{n}. \tag{71}$$
We prove that there is $\gamma>0$ such that
$$1+\frac{1+\beta}{n}>\left(\frac{n+1}{n}\right)^{1+\gamma} \tag{72}$$
for large n.

By MVT we have, for some $\xi\in\left(1,\frac{n+1}{n}\right)$,
$$\left(\frac{n+1}{n}\right)^{1+\gamma}=1+(1+\gamma)\,\xi^{\gamma}\,\frac{1}{n}<1+2^{\gamma}\,\frac{1+\gamma}{n}. \tag{73}$$
Now any γ such that $2^{\gamma}(1+\gamma)<1+\beta$ suffices.

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15. $\lim_{n\rightarrow\infty}\frac{(n+1)^a}{n^a}=1$; $\lim_{n\rightarrow\infty}\left(\frac{1}{n^a}\right)^{1/n}=1$.

16. Joseph Ludwig Raabe, 1801 - 1859.

17. We have
$$n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{n\,x}{n+1}\longrightarrow x. \tag{69}$$
Therefore the series converges if $x>1$ and diverges if $x<1$. When $x=1$ we have $a_n=\frac{1}{n+1}$ so the series also diverges in this case.

Now we have

$$a_n < \left(\frac{n-1}{n}\right)^{1+\gamma} a_{n-1} < \left(\frac{n-2}{n}\right)^{1+\gamma} a_{n-2} < \cdots < \frac{c}{n^{1+\gamma}} \tag{74}$$

and converges follows. □

Exercise 18. Prove the other half: $\sum_{n=1}^{\infty} a_n$ diverges if

$$\limsup_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1. \tag{75}$$

Remark 32. Note that the "convergent" part of Raabe's test can be turned into a convergence test for general series (without requiring $a_n > 0$) as

$$\liminf_{n\longrightarrow\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converges} \tag{76}$$

but the divergent part cannot, that is

$$\limsup_{n\longrightarrow\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges} \tag{77}$$

is not true, for the following reason.

The convergence of $\sum_{n=1}^{\infty} |a_n|$ implies the convergence of $\sum_{n=1}^{\infty} a_n$; But the divergence of $\sum_{n=1}^{\infty} |a_n|$ does not imply the divergence of $\sum_{n=1}^{\infty} a_n$, as the example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ shows

Note that Raabe's Test is sharper than the ratio/root tests, due to the fact that $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges/diverges slower than the geometric series. If we consider even slower convergent series, such as $\sum_{n=1}^{\infty} \frac{1}{n\,(\log(n+1))^a}$, we can obtain even sharper test such as the following.

Theorem 33. (Gauss's Test) [18] *Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series. Further assume we can write*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\beta}{n} + \frac{\theta_n}{n^{1+\kappa}} \tag{78}$$

where $\{\theta_n\}$ is a bounded sequence and $\kappa > 0$. Then

i. If $\beta > 1$, then $\sum_{n=1}^{\infty} a_n$ converges;

ii. If $\beta \leqslant 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. See Theorem 2.12 of (RIS) . Note that the only new situation here is $\beta = 1$. □

Remark 34. It is possible to design even finer tests using the convergence (for $a > 1$)/divergence (for $a \leqslant 1$) of

$$\sum_{n=1}^{\infty} \frac{1}{n\log(n+1)\log(\log(n+1)+1)\cdots\log(\cdots(\log(n+1)+1)\cdots+1)^a}. \tag{79}$$

The results form the so-called Bertrand's Tests. See §2.4 of (RIS) .

18. Karl Friedrich Gauss (1777 - 1855).

4. Advanced Topics, Notes, and Comments

4.1. Abel's formula

Example 35. Consider

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{80}$$

We show that it actually converges. Consider the partial sum

$$S_m = \sum_{k=n+1}^{m} \frac{\sin k}{k}. \tag{81}$$

Denote $A_k = \sum_{l=1}^{k} \sin l$, and $B_k = \frac{1}{k}$. Then we have

$$\begin{aligned}
\sum_{k=n+1}^{m} \frac{\sin k}{k} &= \sum_{k=n+1}^{m} (A_k - A_{k-1}) B_k \\
&= [A_{n+1} B_{n+1} - A_n B_{n+1}] + [A_{n+2} B_{n+2} - A_{n+1} B_{n+2}] + \cdots \\
&\quad + [A_m B_m - A_{m-1} B_m] \\
&= [A_m B_m - A_n B_{n+1}] + \\
&\quad + [A_{n+1} (B_{n+1} - B_{n+2}) + A_{n+2} (B_{n+2} - B_{n+3}) + \cdots + A_{m-1} (B_{m-1} - B_m)] \\
&= [A_m B_m - A_n B_{n+1}] + \sum_{k=n+1}^{m-1} [A_k (B_k - B_{k+1})] \\
&= \left[\frac{A_m}{m} - \frac{A_n}{n+1}\right] + \sum_{k=n+1}^{m-1} \left[A_k \left(\frac{1}{k} - \frac{1}{k+1}\right)\right].
\end{aligned} \tag{82}$$

Now notice that $\{A_n\}$ is in fact a bounded sequence:

$$\begin{aligned}
A_n &= \sin 1 + \sin 2 + \cdots + \sin n \\
&= \frac{\sin 1 [\sin 1 + \sin 2 + \cdots + \sin n]}{\sin 1} \\
&= \frac{\cos (1-1) - \cos (1+1) + \cos (2-1) - \cos (2+1) + \cdots + \cos (n-1) - \cos (n+1)}{2 \sin 1} \\
&= \frac{[\cos 0 + \cos 1 + \cdots + \cos (n-1)] - [\cos 2 + \cos 3 + \cdots + \cos (n+1)]}{2 \sin 1} \\
&= \frac{\cos 0 + \cos 1 - \cos n - \cos (n+1)}{2 \sin 1}.
\end{aligned} \tag{83}$$

Now it is clear that $|A_n| \leqslant \frac{2}{\sin 1}$ for all $n \in \mathbb{N}$.

Back to (82):

$$\begin{aligned}
\left|\sum_{k=n+1}^{m} \frac{\sin k}{k}\right| &\leqslant \left|\frac{A_m}{m}\right| + \left|\frac{A_n}{n+1}\right| + \sum_{k=n+1}^{m-1} |A_k| \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&\leqslant \frac{2}{\sin 1} \left[\frac{1}{m} + \frac{1}{n+1} + \sum_{k=n+1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1}\right)\right] \\
&= \frac{2}{\sin 1} \left[\frac{2}{n+1}\right] = \frac{4}{(n+1) \sin 1}.
\end{aligned} \tag{84}$$

Now we are ready to show the series is Cauchy:

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N + 1 \geqslant \frac{4}{\varepsilon \sin 1}$, then for any $m > n > N$, we have

$$\left| \sum_{k=n+1}^{m} \frac{\sin k}{k} \right| \leqslant \frac{4}{(n+1)\sin 1} < \frac{4}{(N+1)\sin 1} \leqslant \varepsilon. \tag{85}$$

Therefore the series converges.

Exercise 19. Prove the following Abel summation formula: Let $A_k = \sum_{l=1}^{k} a_l$, $B_k = \sum_{l=1}^{k} b_l$, then

$$\sum_{k=n+1}^{m} a_k B_k = [A_m B_m - A_n B_{n+1}] - \sum_{k=n+1}^{m-1} A_k b_{k+1} \tag{86}$$

Draw analogy to the formula of integration by parts:

$$\int_a^b f(x) G(x) \, \mathrm{d}x = [F(b) G(b) - F(a) G(a)] - \int_a^b F(x) g(x) \, \mathrm{d}x \tag{87}$$

where $F(x) = \int_a^x f(t) \, \mathrm{d}t$, $G(x) = \int_a^x g(t) \, \mathrm{d}t$.

Exercise 20. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} \frac{a_n}{n}$. (Hint:[19])

THEOREM 36. *Consider $\sum_{n=1}^{\infty} (-1)^n a_n$ with $a_n \geqslant 0$. If*

i. There is $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $a_{n+1} \leqslant a_n$;

ii. $\lim_{n\to\infty} a_n = 0$;

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges to some $s \in \mathbb{R}$.

Exercise 21. Prove the theorem. (Hint:[20])

4.2. Conditionally convergent series

Recall that a series $\sum_{n=1}^{\infty} a_n$ is called "conditionally convergent" if it is convergent, but $\sum_{n=1}^{\infty} |a_n| = \infty$. Otherwise it is called "absolutely convergent".

Example 37. (GROUPING) Unless $a_n \geqslant 0$ (or all $\leqslant 0$) and $\sum_{n=1}^{\infty} a_n$ converges, the order of summation cannot be changed. For example let $a_n = (-1)^{n+1}$. If we are allowed to group terms together and sum them first, we would have both

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = 1 + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + \cdots = 1; \tag{89}$$

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0. \tag{90}$$

DEFINITION 38. (REARRANGEMENT) *A rearrangement of an infinite series $\sum_{n=1}^{\infty} a_n$ is another infinite series $\sum_{m=1}^{\infty} a_{n(m)}$ where $m: \mapsto n(m)$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$.*

19. Denote $S_n := \sum_1^n a_m$. Then there is $M > 0$ such that for all $n \in \mathbb{N}$, $|S_n| < M$. Now we can show the series is Cauchy through

$$\left| \sum_{k=n+1}^{m} \frac{a_k}{k} \right| = \sum_{k=n+1}^{m} \frac{S_k - S_{k-1}}{k} = \left| -\frac{S_n}{n+1} + \frac{S_m}{m} + \sum_{k=n+1}^{m-1} S_k \left(\frac{1}{k} - \frac{1}{k+1} \right) \right| \leqslant \frac{|S_n|}{n+1} + \frac{|S_m|}{m} + \sum_{k=n+1}^{m-1} |S_k| \left(\frac{1}{k} - \frac{1}{k+1} \right). \tag{88}$$

20. Apply Abel's formula to the partial sums.

Example 39. An example of rearrangement of $a_1+a_2+a_3+\cdots$ is $a_2+a_4+a_7+a_1+a_5+a_3+a_6+\cdots$.

Exercise 22. Prove that if $\sum_{n=1}^{\infty} a_n = s$ is absolutely convergent, then any re-grouping or re-arrangement give the same sum. (Hint:[21])

Example 40. (REARRANGEMENT) Consider the sequence $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{(-1)^{n+1}}{n}$. If we are allowed to freely rearrange (that is choose the order of summation), then for any $s \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a rearrangement such that it converges to s.

Proof. Consider the case $s \in \mathbb{R}$. The cases $s = \infty, -\infty$ are left as exercises.

Consider the rearrangement $\sum_{n=1}^{\infty} b_n$ defined as follows:

- Let k_0 be such that $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 1} \geqslant s$ but $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 3} < s$. Set
$$b_1 = 1, b_2 = \frac{1}{3}, \ldots, b_{k_0} = \frac{1}{2\,k_0 - 1}; \tag{92}$$
The case $k_0 = 1$ is when $1 \geqslant s$. Then we just set $b_1 = 1$ and turn to the next step.
- Let k_1 be such that
$$\sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1 - 2}\right) \geqslant s, \qquad \sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1}\right) < s \tag{93}$$
and set
$$b_{k_0+1} = -\frac{1}{2}, \qquad b_{k_0+k_1} = -\frac{1}{2\,(k_1+1)}. \tag{94}$$
- Let k_2 be such that
$$\sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 1}\right) \geqslant s, \sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 3}\right) < s \tag{95}$$
and set
$$b_{k_0+k_1+1} = \frac{1}{2\,k_0+1}, \ldots, b_{k_0+k_1+k_2} = \frac{1}{2\,k_0 + 2\,k_2 + 1}, \tag{96}$$
- And so on.

Now set
$$S_l = \sum_{k=1}^{k_0+k_1+\cdots+k_l} b_k. \tag{97}$$
Then we see that if $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$, then
$$s_n = \sum_{m=1}^{n} b_m \tag{98}$$

21. For re-grouping, not that the convergence of the re-grouped series is equivalent to the convergence of a subsequence S_{n_k} of the partial sums S_n. The conclusion follows from, for any $n > n_k$,
$$|S_n - S_{n_k}| \leqslant \sum_{m=n_k+1}^{n} |a_m|. \tag{91}$$
For re-arrangement, first prove that any positive convergent series can be re-arranged without changing the sum. Then consider $\sum_{n=1}^{\infty} |a_n| + a_n$ and $\sum_{n=1}^{\infty} |a_n|$.

is always between S_l and S_{l+1}.

Finally notice that by construction, $|S_l - s| < \frac{1}{l}$. Thus for any $\varepsilon > 0$, take $L \in \mathbb{N}$ such that $L > \varepsilon^{-1}$. Now set $N = k_0 + \cdots + k_L$. For any $n > N$, there is $l \geqslant L$ such that $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$. Therefore we have

$$|s_n - s| \leqslant \max\{|S_l - s|, |S_{l+1} - s|\} \leqslant \frac{1}{l} \leqslant \frac{1}{L} < \varepsilon. \tag{99}$$

That is $\sum_{n=1}^{\infty} b_n \longrightarrow s$ by definition. □

Remark 41. Note that the above proof depends on the fact that

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} = \infty, \qquad \sum_{k=1}^{\infty} \left(-\frac{1}{2k}\right) = -\infty. \tag{100}$$

Exercise 23. Prove the above two facts. (Hint:[22])

THEOREM 42. *Let $\sum_{n=1}^{\infty} a_n$ be a series whose terms go to 0. Then exactly one of the following holds:*

- *$\sum_{n=1}^{\infty} a_n$ converges absolutely;*
- *All re-arrangements diverge to ∞;*
- *All re-arrangements diverge to $-\infty$;*
- *The series can be re-arranged to sum to any $r \in \mathbb{R} \cap \{\infty, -\infty\}$.*

Proof. Problem 12. □

4.3. There is no ultimate test

We prove here that a "ultimate" test does not exist. More specifically, we cannot find the "largest convergent series" nor the "smallest divergent series".

THEOREM 43. (RIS GEM 47, 48, CMJ, 28:4, PP. 296 - 297)

a) *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms. Then there exists another convergent series $\sum_{n=1}^{\infty} A_n$ that decreases more slowly in the sense that*

$$\lim_{n \to \infty} \frac{A_n}{a_n} = \infty. \tag{101}$$

b) *Let $\sum_{n=1}^{\infty} D_n$ be a divergent series of positive terms. Then there exists another divergent series $\sum_{n=1}^{\infty} d_n$ that is smaller in the sense that*

$$\lim_{n \to \infty} \frac{D_n}{d_n} = \infty. \tag{102}$$

Proof.

a) Let $r_n := a_n + \cdots$. Then we have $r_n \searrow 0$. Now define

$$A_n := \frac{a_n}{r_n^{\alpha}}. \tag{103}$$

22. Comparison with $\sum \frac{1}{n}$.

for any $\alpha \in (0,1)$. Then we have

$$\sum_{n=1}^{N} A_n = \sum_{n=1}^{N} \frac{1}{r_n^\alpha} (r_n - r_{n+1}) \leqslant \int_0^{r_1} \frac{1}{t^\alpha} \mathrm{d}t. \tag{104}$$

Since this bound is uniform in N, we see that $\sum_{n=1}^\infty A_n$ converges.

b) Let $S_n := D_1 + \cdots + D_n$, then $S_n \longrightarrow \infty$ since $\sum D_n$ diverges. Define $d_n = \frac{D_n}{S_n}$. Then we have

$$\sum_{k=n+1}^{m} d_k \geqslant \sum_{k=n+1}^{m} \frac{D_k}{S_m} = \frac{S_m - S_n}{S_m} = 1 - \frac{S_n}{S_m}. \tag{105}$$

Since $\lim_{n\to\infty} S_n = \infty$, for any $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $\frac{S_n}{S_m} < \frac{1}{2}$. Thus $\sum_{n=1}^\infty d_n$ is not Cauchy and cannot converge. □

4.4. Infinite product

Consider the product of infinitely many real numbers:

$$\prod_{n=1}^{\infty} p_n := p_1 \cdot p_2 \cdot p_3 \cdots. \tag{106}$$

Note the $s \neq 0$!

DEFINITION 44. *We say the infinite product $\prod_{n=1}^\infty p_n$ converges if and only if $P_n = p_1 \cdots p_n$ converges to $s \in \mathbb{R}$, $s \neq 0$. We say $\prod_{n=1}^\infty p_n$ diverges if and only if it does not converge.*

Exercise 24. Prove that $\frac{1}{2} \cdot \frac{1}{2} \cdots$ diverges.

Exercise 25. Prove that, if $\prod_{n=1}^\infty p_n$ converges, then $\lim_{n\to\infty} p_n = 1$. (Hint:[23])

Exercise 26. Prove that, if $\prod_{n=1}^\infty p_n$ converges, then there is $N \in \mathbb{N}$ such that for all $n > N$, $p_n > 0$. (Hint:[24])

Example 45. $\prod_{n=1}^\infty \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Proof. We have

$$P_n = \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right) \cdots = \frac{1}{2}\left(1 + \frac{1}{n}\right). \tag{107}$$

The conclusion follows. □

Exercise 27. Let $|x| < 1$. Prove that

$$\prod_{n=1}^{\infty} (1 + x^{2^{n-1}}) = \frac{1}{1-x}. \tag{108}$$

Now we consider writing $p_n = 1 + a_n$ with $a_n > -1$. From the exercises following Definition 44 we see that this is reasonable.

Exercise 28. Prove that if $\prod_{n=1}^\infty (1 + a_n)$ converges, then $\lim_{n\to\infty} a_n = 0$.

23. Take ln or prove directly.
24. Otherwise P_n "oscillates".

THEOREM 46. *The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *converges if and only if the infinite series* $\sum_{n=1}^{\infty}\ln(1+a_n)$ *converges. Furthermore in this case we have*

$$\prod_{n=1}^{\infty}(1+a_n)=\exp\left[\sum_{n=1}^{\infty}\ln(1+a_n)\right]. \tag{109}$$

Exercise 29. Prove the theorem.

THEOREM 47. *If there is* $N\in\mathbb{N}$ *such that* $\forall n>N$, $a_n>0$, *then*

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges} \Longleftrightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{110}$$

Proof. When $a_n>0$ we have $a_n>\ln(1+a_n)>0$ which gives $\Longleftarrow$.

For $\Longrightarrow$ we notice that the convergence of $\prod_{n=1}^{\infty}(1+a_n)$ gives $\lim_{n\to\infty}a_n=0$ which implies

$$\lim_{n\to\infty}\frac{\ln(1+a_n)}{a_n}=1. \tag{111}$$

From this it is easy to obtain $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges $\Longrightarrow \sum_{n=1}^{\infty}a_n$ converges. $\square$

Exercise 30. Prove that, if there is $N\in\mathbb{N}$ such that $\forall n>N$, $a_n<0$, then

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges} \Longleftrightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{112}$$

DEFINITION 48. *Say* $\prod_{n=1}^{\infty}(1+a_n)$ *absolutely convergent, if and only if* $\prod_{n=1}^{\infty}(1+|a_n|)$ *converges.*

Exercise 31. If $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent, it is convergent.

Exercise 32. If $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent, then any re-arrangement converges to the same value.

4.5. Some fun examples

Example 49. We know that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to ∞. Now consider the sequence obtained by dropping all terms involving 9:

$$1+\frac{1}{2}+\cdots+\frac{1}{8}+\cancel{\frac{1}{9}}+\frac{1}{10}+\cdots+\frac{1}{18}+\cancel{\frac{1}{19}}+\cdots+\frac{1}{88}+\cancel{\frac{1}{89}}+\cancel{\frac{1}{90}}+\cancel{\frac{1}{91}}+\cancel{\frac{1}{92}}+\cdots \tag{113}$$

Does this series converge?

As it is a non-negative sequence, we only need to check whether it's bounded above. We have,

$$1+\cdots+\frac{1}{8}<8, \tag{114}$$

$$\left(\frac{1}{10}+\cdots+\frac{1}{18}\right)+\cdots+\left(\frac{1}{80}+\cdots\frac{1}{88}\right)<8\cdot\frac{9}{10} \tag{115}$$

In general, there are $8\cdot 9^k$ terms between 10^{-k} and $\frac{1}{10^{k+1}-1}$, so their sum is bounded above by $8\cdot\left(\frac{9}{10}\right)^k$. Overall the sum is bounded above by

$$\sum_{n=0}^{\infty}8\cdot\left(\frac{9}{10}\right)^n=8\cdot\frac{1}{1-\frac{9}{10}}=80. \tag{116}$$

Therefore the new series converges.

Remark 50. Obviously we can try to study the sequence resulted from deleting all terms involving other digits, or sequence of numbers. For example we can delete all terms involving the combination 43, that is $\frac{1}{4352}$ is deleted while $\frac{1}{4537}$ is not. We can even play some silly games such as deleting all terms involving 121221, or someone's birthday. The resulting sequences are all convergent.

Example 51. (RIS GEM 87) For each positive integer n, let $f(n)$ be the number of zeros in the decimal representation of n. Choose $a > 0$ and consider the series

$$\sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2}. \tag{117}$$

This series converges for $0 < a < 91$ and diverges for $a \geqslant 91$.

Proof. Fix $k \in \mathbb{N}$. There are exactly $\binom{m}{k} 9^{m+1-k}$ numbers with $f(n) = k$. Now we re-arrange the series such that all numbers with the same number of digits and the same $f(n)$ are summed first. This is OK since the series is positive.

If we sum all numbers with $m+1$ digits, we have

$$\sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2} \geqslant \sum_{k=0}^{m} \binom{m}{k} 9^{m+1-k} \frac{a^k}{10^{2m+2}} = \frac{9}{100} \left(\frac{9+a}{100} \right)^m. \tag{118}$$

Thus we prove the divergence for $a \geqslant 91$.

For the other half, we use

$$\sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2} \leqslant \sum_{k=0}^{m} \binom{m}{k} 9^{m+1-k} \frac{a^k}{10^{2m}} = 9 \left(\frac{9+a}{100} \right)^m \tag{119}$$

and proceed similarly. □

Example 52. (FERMAT'S LAST THEOREM) This is adapted from the blog of Terence Tao of UCLA[25]. We all know that Fermat's Last Theorem claims that

$$x^n + y^n = z^n, \qquad x, y, z \in \mathbb{N} \tag{120}$$

does not have any solution when $n \geqslant 3$. On the other hand, it is well-known that when $n = 2$, there are infinitely many solutions. But why? What's the difference between $n = 2$ and $n > 2$? It turns out that we can reveal some difference through knowledge of convergence/divergence of infinite series.

Let's consider the chance of three numbers $a, b, a+b$ are all the nth power of a natural number. If we treat a as a typical number of size a, then it's chance of being an nth power is roughly $a^{1/n}/a$. Ignoring the relation between $a, b, a+b$, we have the following probability for $a, b, a+b$ solving the equation (120):

$$a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a+b)^{\frac{1}{n}-1}. \tag{121}$$

Now consider all numbers a, b, we sum up the probabilities:

$$I := \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \left[a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a+b)^{\frac{1}{n}-1} \right] \tag{122}$$

25. The probabilistic heuristic justification of the ABC conjecture, link at http://terrytao.wordpress.com/2012/09/18/the-probabilistic-heuristic-justification-of-the-abc-conjecture/

and apply the following intuition based on the so-called Borel-Cantelli Lemma in probability:

> If $I < \infty$, then the chance of (120) having a solution is very low, while if $I = \infty$, the chance is very high.

We notice that I has perfect symmetry between a and b, which means

$$I = 2\sum_{a=1}^{\infty}\sum_{b=1}^{a-1}\left[a^{\frac{1}{n}-1}\,b^{\frac{1}{n}-1}\,(a+b)^{\frac{1}{n}-1}\right] + \sum_{a=1}^{\infty} a^{\frac{2}{n}-2}\,(2\,a)^{\frac{1}{n}-1}. \tag{123}$$

It is clear that the second series converges for all $n \geqslant 2$ so can be ignored for our purpose.

Now consider

$$J = \sum_{a=1}^{\infty}\sum_{b=1}^{a-1}\left[a^{\frac{1}{n}-1}\,b^{\frac{1}{n}-1}\,(a+b)^{\frac{1}{n}-1}\right]. \tag{124}$$

When $1 \leqslant b \leqslant a-1$, we have $a < a+b < 2\,a$ therefore

$$\sum_{a=1}^{\infty}\sum_{b=1}^{a-1} a^{\frac{1}{n}-1}\,b^{\frac{1}{n}-1}\,(2\,a)^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty}\sum_{b=1}^{a-1} a^{\frac{1}{n}-1}\,b^{\frac{1}{n}-1}\,(a)^{\frac{1}{n}-1} \tag{125}$$

which gives

$$2^{\frac{1}{n}-1}\sum_{a=1}^{\infty}\sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\,b^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty}\sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\,b^{\frac{1}{n}-1}. \tag{126}$$

So finally all we need to study is the convergence/divergence of

$$K = \sum_{a=1}^{\infty}\sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\,b^{\frac{1}{n}-1} = \sum_{a=1}^{\infty}\left[a^{\frac{2}{n}-2}\sum_{b=1}^{a-1} b^{\frac{1}{n}-1}\right]. \tag{127}$$

We have

$$\sum_{b=1}^{a-1} b^{\frac{1}{n}-1} \sim \int_1^{a-1} x^{\frac{1}{n}-1}\,\mathrm{d}x \sim a^{\frac{1}{n}}. \tag{128}$$

Therefore

$$K \sim \sum_{a=1}^{\infty} a^{\frac{3}{n}-2} \tag{129}$$

which is convergent when $n \geqslant 4$ while divergent when $n = 2$.

Remark 53. The case $n = 3$ is a bit tricky here. The series $\sum_{a=1}^{\infty} a^{\frac{3}{n}-2}$ becomes the Harmonic series $\sum_{a=1}^{\infty} a^{-1}$ which is the borderline between convergence and divergence. Our argument does not provide any insight on why $x^3 + y^3 = z^3$ should not have solutions.

Exercise 33. (RIS, §6.1, PUZZLE 3) Desert Dan is trying to cross a 500-mile desert in a jeep. Unfortunately, his jeep could hold only enough fuel to travel 50 miles. However, Desert Dan has an infinite collection of containers that he can use to leave caches of fuel for himself to use on a later trip. Further, on the entering edge of the desert there is a boundless supply of fuel, so that Desert Dan can return to the entering edge of the desert as often as he wishes. Can he cross the desert? (Hint: Footnote[26].)

26. Design a strategy so that in n trips Desert Dan can travel $25\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$ miles.

5. More exercises and problems

Note. Many of the following problems are from (RIS) Chapters 4 and 5. Please note that I chose those problems that are not particularly tricky and may help understanding concepts and establishing intuitions. Please check out (RIS) if you would like to challenge your brain power.

5.1. Basic exercises

5.1.1. Convergence of infinite series: Definition and properties

Exercise 34. Find a divergent series $\sum_{n=1}^{\infty} a_n$ such that for every fixed $k \in \mathbb{N}$, $\lim_{n \to \infty} (a_n + \cdots + a_{n+k}) = 0$. (Hint:[27])

Exercise 35. Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Let $\sum_{n=1}^{\infty} b_n$ be the series obtained from a_n by deleting all the zeroes. (For example $1 + 0 + 3 + 2 + 0 + 5 + \cdots$ would be come $1 + 3 + 2 + 5 + \cdots$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ convergens, and when converge they have the same sum. (Hint: See this footnote.[28])

Exercise 36. Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Let $(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots$ be any grouping of the series. Prove that if this new series does not converge some finite number, then $\sum_{n=1}^{\infty} a_n$ does not converge either. (Hint:[29])

Exercise 37. (RIS) Prove

$$\sum_{k=0}^{\infty} \frac{k^2 + 3\,k + 1}{(k+2)!} = 2. \tag{130}$$

(Hint:[30])

5.1.2. Comparison

Exercise 38. Assume $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} \frac{(n+1)^2\, a_n\, b_n}{n^2}$ converges.

Exercise 39. Assume $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge. Prove that $\sum_{n=1}^{\infty} \frac{(n+1)^2\, a_n\, b_n}{n^2}$ converges. (Hint:[31])

Exercise 40. (Putnam 1940) Assume $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Let $p \geqslant 2$. Prove that $\sum_{n=1}^{\infty} |a_n - b_n|^p$ converges. (Hint:[32])

Exercise 41. (RIS) If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are non-negative and convergent, then $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ is also convergent. Does the claim still hold if non-negativity is dropped from either a_n or b_n? (Hint:[33])

Exercise 42. (RIS) Suppose $\sum_{n=1}^{\infty} a_n$ convergens but $\sum_{n=1}^{\infty} a_n^2$ diverges. Prove that $\sum |a_n|$ diverges.

Exercise 43. (Ratio comparison test) Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be positive series. Assume there is $N_0 \in \mathbb{N}$ such that $\forall n > N_0$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$. Then

i. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges;

ii. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Exercise 44. Prove or disprove: If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \tan(a_n)$ converges. (Hint:[34])

Exercise 45. Prove that

$$\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1} \right) \tag{131}$$

27. $a_n = 1/n$.

28. The partial sums of $\sum b_n$ is a subsequence of that of $\sum a_n$.

29. Cauchy criterion (Or definition, notice that the new partial sums form a subsequence of the partial sums of $\sum a_n$).

30. $k^2 + 3\,k + 1 = (k+2)\,(k+1) - 1$.

31. $a_n, b_n \longrightarrow 0$ therefore $a_n^2 \leqslant |a_n|, b_n^2 \leqslant |b_n|$ for large n.

32. If $|a_n - b_n| \leqslant 1$ then $|a_n - b_n|^p \leqslant |a_n - b_n|^2$.

33. No. Consider $a_n = \frac{(-1)^n}{n}$, $b_n = 0$.

34. $a_n \longrightarrow 0$. Comparison.

converges. (Hint:[35])

5.1.3. Tests

Exercise 46. (PUTNAM 1942) Is the following series convergent or divergent?

$$1+\frac{1}{2}\cdot\frac{19}{7}+\frac{2!}{3^2}\left(\frac{19}{7}\right)^2+\frac{3!}{4^3}\left(\frac{19}{7}\right)^3+\frac{4!}{5^4}\left(\frac{19}{7}\right)^4+\cdots \tag{132}$$

(Hint:[36])

5.1.4. Abel's re-summation

Exercise 47. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges. (Hint: [37].)

Exercise 48. Let λ be any positive number. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} n\, e^{-\lambda n}\, a_n$ also converges. (Hint:[38])

Exercise 49. Prove the following Dirichlet's Test:

> Consider $\sum_{n=1}^{\infty} a_n\, b_n$. Assume that b_n decreases to 0 and the partial sums of $\sum a_n$ are uniformly bounded (in n). Then $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

Exercise 50. Prove the following Abel's Test:

> Let $\sum_{n=1}^{\infty} a_n$ be convergent and $\{b_n\}$ positive and decreasing ($\lim_{n\to\infty} b_n$ may not be 0). Then $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

5.2. More exercises

Exercise 51. Let $\sum_{n=1}^{\infty} a_n$ be positive and decreasing. Then $\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow \lim_{n\to\infty} n\, a_n = 0$. Further prove that "positive" and "decreasing" are both necessary. Also prove that the converse is false. (Hint:[39])

Exercise 52. Let $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ convergent. Set $r_n := \sum_{k=n}^{\infty} a_k$. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges. (Hint:[40])

Exercise 53. (RIS GEM 21, CMJ, 16:2, P, 79) The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{2\,n-1}{2\,n}\right)^p \tag{133}$$

converges if and only if $p > 2$. (Hint: See this footnote.[41])

Exercise 54. (RIS) $\sum_{n=1}^{\infty} (1/n^{1+1/n})$ diverges. (Hint: See footnote[42].)

Exercise 55. (PUTNAM 1950) Study the convergence/divergence of the following series

$$\frac{1}{\log(2!)}+\cdots+\frac{1}{\log(n!)}+\cdots; \qquad \frac{1}{3}+\frac{1}{3\, 3^{1/2}}+\frac{1}{3\cdot 3^{1/2}\cdot 3^{1/3}}+\cdots \tag{134}$$

(Hint: Footnote.[43])

Exercise 56. (RIS) For any positive integers n and p, we have

$$\sum_{m=n}^{\infty}\frac{1}{m\,(m+1)\cdots(m+p)}=\frac{1}{p\,n\,(n+1)\cdots(n+p-1)}. \tag{135}$$

(Hint:[44])

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35. MVT.
36. Ratio test.
37. Abel resummation.
38. $n\, e^{-\lambda n}$ is decreasing for large n.
39. $\sum_{n\geqslant N} a_n \geqslant (m-N)\, a_m$ for every $m > N$.
40. Cauchy criterion.
41. Set $b_n = \left(\frac{2}{3}\,\frac{4}{5}\cdots\frac{2\,n-2}{2\,n-1}\right)^p$. Prove that $\frac{a_n\, b_n}{2} < a_n^2 < a_n\, b_n$.
42. Compare with a divergent series.
43. First one: compare with $\frac{1}{n\log n}$; Second one, $1+\cdots+1/n \sim \log n$.

Exercise 57. (RIS) Proce that

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots=\infty \tag{136}$$

but

$$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\cdots=\log 3. \tag{137}$$

(Hint:[45])

Exercise 58. Let $\sum_{n=1}^{\infty} b_n$ be a positive series. If $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} \frac{b_n}{1+b_n}$. (Hint: Footnote.[46])

Exercise 59. Prove that $\sum_{n=1}^{\infty} (n^{1/n}-1)=\infty$. (Hint: Footnote.[47])

Exercise 60. Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Let $b_n=\begin{cases} a_n & a_n>0 \\ 0 & a_n \leqslant 0 \end{cases}$ and $c_n=\begin{cases} 0 & a_n \geqslant 0 \\ -a_n & a_n<0 \end{cases}$. Prove that

$$\sum_{n=1}^{\infty} b_n=\infty; \qquad \sum_{n=1}^{\infty} c_n=\infty. \tag{138}$$

(Hint:[48])

Exercise 61. (PUTNAM 1956) Given $T_1=2$, $T_{n+1}=T_n^2-T_n+1$, $n>0$. Prove $\sum_{n=1}^{\infty} \frac{1}{T_n}=1$. (Hint: Footnote[49].)

Exercise 62. (PUTNAM 1988) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$(Note that this gives another proof of the fact that there can be not “largest” convergent series) (Hint: [50])

Exercise 63. (FOLLAND) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $\sum_{n=1}^{\infty} b_n$ be such that

$$b_1=a_2, b_2=a_1, b_3=a_4, b_4=a_3, \ldots. \tag{139}$$

Then $\sum_{n=1}^{\infty} b_n$ also converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.

Exercise 64. (FOLLAND) Suppose $a_n>-1$ for all n.

a) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} \ln(1+a_n)$ is absolutely convergent.

b) Find $\sum_{n=1}^{\infty} a_n$ conditionally convergent but $\sum_{n=1}^{\infty} \ln(1+a_n)$ diverges.

(Hint:[51])

Exercise 65. (PUTNAM 1994) Prove that if a_n satisfy $0<a_n<a_{2n}+a_{2n+1}$ for all $n \geqslant 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. (Hint: Footnote[52].) Note that this almost gives another proof of divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$.

Exercise 66. (CAUCHY'S CONDENSATION TEST) Let $\sum_{n=1}^{\infty} a_n$ be positive, and assume there is $N_0 \in \mathbb{N}$ such that $\forall n>N_0, a_{n+1} \leqslant a_n$. Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} 2^n a_{2^n} \tag{140}$$

either both converge or both diverge. (Hint:[53])
Apply Cauchy's Condensation Test to $\sum_{n=1}^{\infty} (1/n^a)$.

44. $\frac{p}{m(m+1)\cdots(m+p)}=\frac{(m+p)-m}{m(m+1)\cdots(m+p)}$.

45. Group 3 terms together.

46. Discuss two cases b_n bounded or not.

47. $\exp\left[\frac{\log n}{n}\right]-\exp 0$. MVT.

48. Assume otherwise. Then either one of $\sum b_n, \sum c_n$ converges, or both converge. In the latter case $\sum a_n$ is absolutely convergent, in the former case $\sum a_n$ cannot be convergent.

49. $\frac{1}{T_{n+1}-1}=\frac{1}{T_n-1}-\frac{1}{T_n}$.

50. Prove $a_n^{n/(n+1)} \leqslant 2a_n+2^{-n}$.

51. For a), first notice $a_n \longrightarrow 0$. Then control the ratio between $|a_n|$ and $|\ln(1+a_n)|$; For b), pick a_n such that $\sum a_n^2$ diverges.

52. Prove that if $a_1 \neq 0$ then the sequence cannot be Cauchy.

53. We have

$$\sum_{2^n}^{2^{n+1}-1} a_k \leqslant 2^n a_{2^n} \leqslant 2 \sum_{2^{n-1}}^{2^n-1} a_k. \tag{141}$$

Exercise 67. (KUMMER) [54]Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, and let $\{\lambda_n\}$ be any sequence of positive numbers. If there is $N_0 \in \mathbb{N}$ such that

$$\forall n > N_0, \qquad \lambda_n - \frac{a_{n+1}}{a_n} \lambda_{n+1} \geqslant k > 0 \tag{142}$$

for some constant k, then $\sum_{n=1}^{\infty} a_n$ converges. (Hint:[55])

Exercise 68. (KUMMER'S TEST) Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series, let $\sum_{n=1}^{\infty} \frac{1}{d_n}$ be positive and divergent, define

$$\kappa_n := d_n - \frac{a_{n+1}}{a_n} d_{n+1}. \tag{143}$$

Then

- if there is $N_0 \in \mathbb{N}$ such that $\forall n > N_0, \kappa_n > k > 0$ for some constant k, then $\sum_{n=0}^{\infty} a_n$ converges.
- if there is $N_0 \in \mathbb{N}$ such that $\forall n > N_0, \kappa_n \leqslant 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Exercise 69. Write Kummer's test in the limit form using limsup and liminf.

Exercise 70. (USTC3) Let $\prod_{n=1}^{\infty} p_n$ and $\prod_{n=1}^{\infty} q_n$ be convergent infinite products. Discuss the convergence of the following infinite products:

$$\prod_{n=1}^{\infty} (p_n + q_n), \quad \prod_{n=1}^{\infty} (p_n - q_n), \quad \prod_{n=1}^{\infty} p_n q_n, \quad \prod_{n=1}^{\infty} \frac{p_n}{q_n}. \tag{144}$$

Exercise 71. (USTC3) Prove the following

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}; \quad \prod_{n=2}^{\infty} \left[1 - \frac{2}{n(n+1)}\right] = \frac{1}{3}; \quad \prod_{n=0}^{\infty} \left[1 + \left(\frac{1}{2}\right)^{2^n}\right] = 2; \quad \prod_{n=1}^{\infty} \cos\frac{x}{2^n} = \frac{\sin x}{x}; \tag{145}$$

$$\frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots = \frac{\pi}{2}. \tag{146}$$

Exercise 72. (USTC3) Discuss the convergence of the following infinite products.

$$\prod_{n=1}^{\infty} \frac{1}{n}; \quad \prod_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}; \quad \prod_{n=2}^{\infty} \left(\frac{n^2-1}{n^2+1}\right)^p; \quad \prod_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}; \quad \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{1/n}; \quad \prod_{n=1}^{\infty} \left[\ln\frac{n+x}{n}\right]^{1/n}. \tag{147}$$

5.3. Problems

Problem 1.

a) (PUTNAM 1964) Prove that there is a constant K such that the following inequality holds for any sequence of positive numbers $a_1, a_2, \ldots$:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leqslant K \sum_{n=1}^{\infty} \frac{1}{a_n}. \tag{148}$$

(Hint: [56].)

b) In the above exercise, consider $a_n = n^k$. We can easily show that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_n}{\frac{1}{k+1} n^{k+1}} = 1$. This would give

$$\frac{n}{a_1 + \cdots + a_n} \sim \frac{k+1}{a_n}. \tag{149}$$

Does this contradict the conclusion of the above exercise? (Hint:[57])

Problem 2. (PUTNAM 1966) Show that if the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges, where $p_n \in \mathbb{R}^+$, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + \cdots + p_n)^2} p_n \tag{150}$$

54. Eduard Kummer (1810 - 1893).

55. $\frac{1}{k} (a_n \lambda_n - a_{n+1} \lambda_{n+1}) \geqslant a_n$

56. Basically, $a_1 + \cdots + a_{2n-1} \geqslant n b_n$ where b_n is the median of $a_1, \ldots, a_{2n-1}$.

57. No. Because (149) only holds when n is large.

is also convergent. (Hint: [58].)

Problem 3. Consider the re-arrangement of $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{(-1)^{n+1}}{n}$ into p positive terms, then q negative terms, then p positive terms, and so on. Then the sum is

$$\ln 2 + \frac{1}{2} (\ln p - \ln q). \tag{151}$$

Problem 4. Prove that for $x_n > 0$

$$\liminf_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n}. \tag{152}$$

(Hint:[59])

Problem 5. (SCHLOMILCH) [60]Let $a_n > 0$ and assume there is $N_0 \in \mathbb{N}$ such that $\forall n > N_0, a_{n+1} \leqslant a_n$. Let $n_1 < n_2 < \cdots$ be a strictly increasing sequence of positive integers such that $(n_{k+1} - n_k)/(n_k - n_{k-1})$ is bounded (as a function of k). Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{k=1}^{\infty} (n_{k+1} - n_k)\, a_{n_k} \tag{153}$$

either both converge or both diverge.

Then apply this result to $\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}}$.

Problem 6. Define $H_n := \sum_{k=1}^{n} \frac{1}{k}$. Write

$$\gamma_n := H_n - \ln n - \frac{1}{n}. \tag{154}$$

Then $\lim_{n \to \infty} \gamma_n$ exists. Then use this to prove $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

Problem 7. Let $a_n > 0$. Is it possible to obtain a "Convergence Test" by studying

$$\limsup_{n \to \infty} \frac{a_{2n} + a_{2n+1}}{a_n} \text{ and } \liminf_{n \to \infty} \frac{a_{2n} + a_{2n+1}}{a_n}? \tag{155}$$

This is inspired by Exercise 65.

Problem 8. Discuss the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+a_n}} \tag{156}$$

where $a_n > 0$.

Problem 9. Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Let $n_p := \min\{n \in \mathbb{N}, H_n > p\}$. Then

$$\lim_{p \to \infty} \frac{n_{p+1}}{n_p} = e. \tag{157}$$

(Solution: Gem 55 of (RIS)).

Problem 10. Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Let $r \in \mathbb{R}$ be arbitrary. Then there is a function f: $\mathbb{N} \mapsto \{0, 1\}$ such that $\sum_{n=1}^{\infty} (-1)^{f(n)} a_n = r$.

Problem 11. Prove the following variant of Gauss's Test:

Let $\sum_{n=1}^{\infty} a_n$ be a positive infinite series. Assume that it satisfies

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\beta}{n \ln n} + R_n \tag{158}$$

where $\lim_{n \to \infty} (n \ln n)\, R_n = 0$. Then the series converges if $\beta > 1$ and diverges if $\beta < 1$. (Hint:[61])

Problem 12. Prove Theorem 42:

58. $H_n := p_1 + \cdots + p_n$. And use $H_n^2 > H_n\, H_{n-1}$. Derive an quadratic inequality for the partial sum through Cauchy-Schwarz.

59. $x_n = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_2}{x_1} x_1$.

60. Oscar Xavier Schl"omilch (1823 - 1901).

61. Try to prove

$$\frac{a_n}{a_{n+1}} - \frac{n+1}{n} \left(\frac{\ln (n+1)}{\ln n} \right)^{\alpha} > 0 \tag{159}$$

for some $\alpha > 0$ and n large enough.

Let $\sum_{n=1}^{\infty} a_n$ be a series whose terms go to 0. Then exactly one of the following holds:

- $\sum_{n=1}^{\infty} a_n$ converges absolutely;
- All re-arrangements diverge to ∞;
- All re-arrangements diverge to $-\infty$;
- The series can be re-arranged to sum to any $r \in \mathbb{R} \cap \{\infty, -\infty\}$.

Problem 13. Assume $\sum_{n=1}^{\infty} a_n^2$ converges. Then

$$\prod_{n=1}^{\infty} (1+a_n) \text{ converges} \Longleftrightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{160}$$

Note that there is no assumption on the sign of a_n. (Hint: [62])

Problem 14. (EULER) In his 1734 paper *De progressionibus harmonicis observationes* (Observations on harmonic progressions)[63], Leonhard Euler did the following manipulations on harmonic-type series. Comment on his methods and results. If the results are right while the methods are not, give correct proofs of the results.

a) Let $a, b, c > 0$. Let $i = \infty$. Denote

$$s = \frac{c}{a} + \frac{c}{a+b} + \frac{c}{a+2\,b} + \cdots + \frac{c}{a+i\,b}. \tag{161}$$

Now we have

$$\frac{\delta s}{\delta i} = \frac{c}{a+i\,b}. \tag{162}$$

Integrate and take $i = \infty$ we have

$$s = C + \frac{c}{b} \ln(a+i\,b). \tag{163}$$

Now replacing i by $n\,i$ and using the fact that

$$\frac{a+i\,b}{a+n\,i\,b} = \frac{a/i+b}{a/i+n\,b} = \frac{0+b}{0+n\,b} = \frac{1}{n} \tag{164}$$

we have

$$\ln(n) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n\,i}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{i}\right). \tag{165}$$

Now re-arrange. we have

$$\ln(n) = 1 + \frac{1}{2} + \cdots + \left(\frac{1}{n} - 1\right) + \frac{1}{n+1} + \cdots + \left(\frac{1}{2\,n} - \frac{1}{2}\right) + \cdots \tag{166}$$

In particular:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots; \tag{167}$$

$$\ln 4 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \cdots \tag{168}$$

Now $2\ln 2 - \ln 4 = 0$ gives

$$0 = 1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \cdots. \tag{169}$$

b) Continuing, Euler obtained:

$$1 = \ln\left(\frac{2}{1}\right) + \frac{1}{2} - \frac{1}{3} + \cdots \tag{170}$$

$$\frac{1}{2} = \ln\left(\frac{3}{2}\right) + \frac{1}{2\cdot 4} - \frac{1}{3\cdot 8} + \cdots \tag{171}$$

$$\frac{1}{3} = \ln\left(\frac{4}{3}\right) + \frac{1}{2\cdot 9} - \frac{1}{3\cdot 27} + \cdots \tag{172}$$

$$\vdots$$

Problem 15. There are more than one ways to assign a number s to an infinite sum $a_1 + a_2 + \cdots$. The above is the most popular one. The following is another way.

62. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges $\Longrightarrow \sum_{n=1}^{\infty} [a_n - \ln(1+a_n)]$ converges.
63. Comm. Acad. Sci. Imp. Petropol. 7 (1734/5) 1740, 150-161.

Let $a_1, \ldots, a_n, \ldots \in \mathbb{R}$. Define

$$s = \sum_{n=1}^{\infty} a_n \tag{173}$$

if and only if for every $\varepsilon > 0$, there is a finite subset $I \subset \mathbb{N}$, such that for every finite subset $\mathbb{N} \supset J \supseteq I$, we have

$$\left| s - \sum_{j \in J} a_j \right| < \varepsilon. \tag{174}$$

a) Prove that if $s = \sum_{n=1}^{\infty} a_n$ by this definition, then $s = \sum_{n=1}^{\infty} a_n$ by Definition 7;

b) Find $\sum_{n=1}^{\infty} a_n$ that is convergent by Definition 7 but not convergent by the above definition;

c) Prove that if $s = \sum_{n=1}^{\infty} a_n$ by the above definition, then s is also the sum of any rearrangement of $\{a_n\}$.

d) Prove that if $s = \sum_{n=1}^{\infty} a_n$ by Definition 7, and $\forall n \in \mathbb{N}, a_n \geqslant 0$, then s is also the sum of any rearrangement of $\{a_n\}$.

e) Find $\sum_{n=1}^{\infty} a_n$ that is convergent by Definition 7, but after rearrangement converge to a different sum.

f) Prove that if $s = \sum_{n=1}^{\infty} a_n$ by the above definition, then any re-grouping gives the same sum. More specifically, if $\cup_{i \in I} S_i = \mathbb{N}$ and $i \neq j \Longrightarrow S_i \cap S_j = \varnothing$, then

$$\sum_{n=1}^{\infty} a_n = \sum_{i \in I} \left(\sum_{k \in S_i} a_k \right) \tag{175}$$

where all three sums are defined as in this problem.

g) Prove that $\sum_{n=1}^{\infty} a_n$ is convergent by the above definition if and only if $\sum_{n=1}^{\infty} |a_n|$ is convergent by Definition 7

This new definition is closely related to the concept of "net", which can assign limits to objects without a "sequence" structure, for example the convergence of Riemann sums of a function to its Riemann integral is convergence in the sense of net.

Remark. When we use the definition in the above problem, we often write $\sum_{n \in \mathbb{N}} a_n$ instead of $\sum_{n=1}^{\infty} a_n$ to emphasize that order does not matter here. $\sum_{n \in \mathbb{N}} a_n$ is called "unordered series".

It is clear that the above definition for convergence of unordered series can be easily extended to the sum of uncountably many numbers. However it turns out that for uncountably many numbers to have a finite sum, at most countably many can be nonzero.