Math 317 Winter 2014 Homework 6 Solutions

Due Mar. 26 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *In the following $a > 0$ is a constant.*

a) *Calculate*

$$\int_L |x\,y|\,\mathrm{d}s \tag{1}$$

with L: $\begin{pmatrix} a\cos^3 t \\ a\sin^3 t \end{pmatrix}$, $t \in [0, 2\pi]$.

b) *Calculate*

$$\int_S (x+y+z)\,\mathrm{d}S \tag{2}$$

with S the first octant part of the unit sphere: $x^2+y^2+z^2 = a^2$, $x, y, z \geqslant 0$.

Solution.

a) We have

$$\begin{aligned}
\int_L |x\,y|\,\mathrm{d}s &= \int_0^{2\pi} |(a\cos^3 t)\,(a\sin^3 t)|\,\sqrt{(-3\,a\cos^2 t\sin t)^2 + (3\,a\sin^2 t\cos t)^2}\,\mathrm{d}t \\
&= \int_0^{2\pi} a^2\,|\cos t\sin t|^3 \sqrt{9\,a^2\cos^4 t\sin^2 t + 9\,a^2\sin^4 t\cos^2 t}\,\mathrm{d}t \\
&= \int_0^{2\pi} a^2\,|\cos t\sin t|^3\,(3\,a)\,|\cos t\sin t| \\
&= 3\,a^3 \int_0^{2\pi} (\cos t\sin t)^4\,\mathrm{d}t \\
&= \frac{3}{16}\,a^3 \int_0^{2\pi} (\sin 2\,t)^4\,\mathrm{d}t \\
&= \frac{3}{16}\,a^3 \int_0^{2\pi} \left(\frac{1-\cos 4\,t}{2}\right)^2 \mathrm{d}t \\
&= \frac{3\,\pi}{32}\,a^3 + \frac{3}{64}\,a^2 \int_0^{2\pi} (\cos 4\,t)^2\,\mathrm{d}t \\
&= \frac{9\,\pi}{64}\,a^3.
\end{aligned} \tag{3}$$

b) We parametrize S:

$$\begin{pmatrix} a\cos\varphi\cos\psi \\ a\sin\varphi\cos\psi \\ a\sin\psi \end{pmatrix}, \qquad \varphi \in \left[0, \frac{\pi}{2}\right], \qquad \psi \in \left[0, \frac{\pi}{2}\right]. \tag{4}$$

Now calculate

$$\boldsymbol{r}_\varphi = a \begin{pmatrix} -\sin\varphi\cos\psi \\ \cos\varphi\cos\psi \\ 0 \end{pmatrix}, \qquad \boldsymbol{r}_\psi = a \begin{pmatrix} -\cos\varphi\sin\psi \\ -\sin\varphi\sin\psi \\ \cos\psi \end{pmatrix}, \tag{5}$$

we have

$$E = a^2\,(\cos\psi)^2, \qquad F = 0, \qquad G = a^2 \Longrightarrow \sqrt{E\,G - F^2} = a^2\,|\cos\psi| = a^2\cos\psi. \tag{6}$$

Thus we have

$$\begin{aligned}\int_S (x+y+z)\,\mathrm{d}S &= \int_{[0,\frac{\pi}{2}]^2} a\,[\cos\varphi\cos\psi+\sin\varphi\cos\psi+\sin\psi]\,a^2\cos\psi\,\mathrm{d}(\varphi,\psi)\\ &= a^3\int_0^{\pi/2}\left[\int_0^{\pi/2}(\cos\varphi+\sin\varphi)\,(\cos\psi)^2+\cos\psi\sin\psi\,\mathrm{d}\psi\right]\mathrm{d}\varphi\\ &= a^3\int_0^{\pi/2}\frac{\pi}{4}(\cos\varphi+\sin\varphi)+\frac{1}{2}\,\mathrm{d}\varphi\\ &= \frac{3\,\pi\,a^3}{4}. \end{aligned} \tag{7}$$

Question 2. *Let $a>0$ be a constant. Calculate*

$$\int_L y\,\mathrm{d}x - z\,\mathrm{d}y + x\,\mathrm{d}z \tag{8}$$

where L is the intersection of $(x^2+y^2)/2+z^2=a^2$ and $x=y$, oriented counter-clockwise when viewed from the positive x-axis.

a) directly;

b) via Stokes's Theorem.

Solution.

a) We notice that on L, $x=y=t$, $t^2+z^2=a^2$. Thus we parametrize L as

$$(a\cos\theta, a\cos\theta, a\sin\theta), \qquad \theta\in[0,2\,\pi]. \tag{9}$$

Note that θ is the angle from the x-y plane to the point on L, therefore the orientation $0\longrightarrow 2\,\pi$ is consistent with the specified orientation.

Thus we have

$$\begin{aligned}\int_L y\,\mathrm{d}x - z\,\mathrm{d}y + x\,\mathrm{d}z &= \int_0^{2\pi}[a\cos\theta\,(a\cos\theta)' - (a\sin\theta)\,(a\cos\theta)' + (a\cos\theta)\,(a\sin\theta)']\,\mathrm{d}\theta\\ &= a^2\int_0^{2\pi}[-\cos\theta\sin\theta+(\sin\theta)^2+(\cos\theta)^2]\,\mathrm{d}\theta\\ &= 2\,\pi\,a^2.\end{aligned} \tag{10}$$

b) We calculate

$$\nabla\times\begin{pmatrix} y\\ -z\\ x\end{pmatrix}=\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}. \tag{11}$$

We pick S to be the part of $x=y$ inside $(x^2+y^2)/2+z^2=a^2$. Then we have

$$\boldsymbol{n}=\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\\ 0\end{pmatrix}. \tag{12}$$

Thus we have

$$\left[\nabla\times\begin{pmatrix} y\\ -z\\ x\end{pmatrix}\right]\cdot\boldsymbol{n}=\sqrt{2}. \tag{13}$$

Note that S is an ellipse with major and minor axes $2\sqrt{2}a$ and $2\,a$, therefore

$$\int_S\left[\nabla\times\begin{pmatrix} y\\ -z\\ x\end{pmatrix}\right]\cdot\boldsymbol{n}\,\mathrm{d}S=\sqrt{2}\,\pi\,\sqrt{2}\,a\,a=2\,\pi\,a^2. \tag{14}$$

Question 3. *Calculate*

$$\int_S \begin{pmatrix} x^2 \\ -y^2 \\ z^2 \end{pmatrix} \cdot \mathbf{d}\boldsymbol{S} \tag{15}$$

where $S = \partial V$ *where* $V = \{x^2 + y^2 + z^2 \leqslant 3\} \cap \{z \geqslant 0\} \cap \left\{z \geqslant \sqrt{x^2 + y^2 - 1}\right\}$, *oriented by the outer normal,*

a) directly;

b) via Gauss's Theorem.

Solution. We can visualize V as a muffin.

a) $S = S_{\text{bottom}} + S_{\text{side}} + S_{\text{top}}$ with

- Bottom: $x^2 + y^2 \leqslant 1, z = 0$;
- Side: $1 \leqslant x^2 + y^2 \leqslant 2, z = \sqrt{x^2 + y^2 - 1}$;
- Top: $x^2 + y^2 \leqslant 2, z = \sqrt{3 - x^2 - y^2}$.

We calculate the integral on each one by one.

- Bottom.

  The natural parametrization is $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$; the outer normal is $\boldsymbol{n}_{\text{bottom}} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Thus

$$\int_{S_{\text{bottom}}} = \int_{u^2+v^2 \leqslant 1} \begin{pmatrix} u^2 \\ -v^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = 0. \tag{16}$$

- Side. We parametrize the side as $\begin{pmatrix} u \\ v \\ \sqrt{u^2 + v^2 - 1} \end{pmatrix}$ with $D = \{1 \leqslant u^2 + v^2 \leqslant 2\}$. Then calculate

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ \frac{u}{\sqrt{u^2 + v^2 - 1}} \end{pmatrix}, \qquad \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ \frac{v}{\sqrt{u^2 + v^2 - 1}} \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} -\frac{u}{\sqrt{u^2 + v^2 - 1}} \\ -\frac{v}{\sqrt{u^2 + v^2 - 1}} \\ 1 \end{pmatrix}. \tag{17}$$

  Note that the outer normal should point down therefore we should use $-\boldsymbol{r}_u \times \boldsymbol{r}_v$ and have

$$\int_{S_{\text{side}}} = \int_D \begin{pmatrix} u^2 \\ -v^2 \\ u^2 + v^2 - 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{u}{\sqrt{u^2 + v^2 - 1}} \\ \frac{v}{\sqrt{u^2 + v^2 - 1}} \\ -1 \end{pmatrix} \mathrm{d}(u, v). \tag{18}$$

  Noticing D is symmetric in u, v we see that

$$\begin{aligned} \int_{S_{\text{side}}} &= \int_D \frac{u^3}{\sqrt{u^2 + v^2 - 1}} \mathrm{d}(u, v) \\ &\quad - \int_D \frac{v^2}{\sqrt{u^2 + v^2 - 1}} \mathrm{d}(u, v) \\ &\quad + \int_D (1 - u^2 - v^2) \mathrm{d}(u, v) \\ &= 0 - 0 + 2\pi \int_1^{\sqrt{2}} (1 - r^2)\, r\, \mathrm{d}r \\ &= -\frac{\pi}{2}; \end{aligned} \tag{19}$$

- Top. We parametrize $\begin{pmatrix} u \\ v \\ \sqrt{3-u^2-v^2} \end{pmatrix}$ with $D=\{u^2+v^2 \leqslant 2\}$. Then calculate

$$\boldsymbol{r}_u = \begin{pmatrix} 1 \\ 0 \\ -\frac{u}{\sqrt{3-u^2-v^2}} \end{pmatrix}, \quad \boldsymbol{r}_v = \begin{pmatrix} 0 \\ 1 \\ -\frac{v}{\sqrt{3-u^2-v^2}} \end{pmatrix} \Longrightarrow \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{pmatrix} \frac{u}{\sqrt{3-u^2-v^2}} \\ \frac{v}{\sqrt{3-u^2-v^2}} \\ 1 \end{pmatrix}. \tag{20}$$

This time it is consistent with the outer normal. Again noticing the symmetry, we have

$$\begin{aligned} \int_{S_{\text{top}}} &= \int_D (3-u^2-v^2)\,\mathrm{d}(u,v) \\ &= 2\pi \int_0^{\sqrt{2}} (3-r^2)\, r\, \mathrm{d}r \\ &= 4\pi. \end{aligned} \tag{21}$$

Adding everything up, we have the original integral to be $7\pi/2$.

b) We calculate

$$\operatorname{div}\begin{pmatrix} x^2 \\ -y^2 \\ z^2 \end{pmatrix} = 2x - 2y + 2z. \tag{22}$$

As V is symmetric in both x, y, we have

$$\int_V (2x-2y+2z)\,\mathrm{d}(x,y,z) = 2\int_V z\,\mathrm{d}(x,y,z). \tag{23}$$

On the other hand,

$$\begin{aligned} V &= \left\{(x,y,z)|\, x^2+y^2+z^2 \leqslant 3, z \geqslant \max\left\{0, \sqrt{x^2+y^2-1}\right\}\right\} \\ &= \left\{x^2+y^2 \leqslant 1, 0 \leqslant z \leqslant \sqrt{3-x^2-y^2}\right\} \cup \left\{1 \leqslant x^2+y^2 \leqslant 2, \sqrt{x^2+y^2-1} \leqslant z \leqslant \sqrt{3-x^2-y^2}\right\} \\ &=: V_1 \cup V_2. \end{aligned}$$

Thus we have

$$\begin{aligned} 2\int_V z\,\mathrm{d}(x,y,z) &= 2\int_{V_1} z\,\mathrm{d}(x,y,z) + \int_{V_2} z\,\mathrm{d}(x,y,z) \\ &= 2\int_{x^2+y^2 \leqslant 1} \left[\int_0^{\sqrt{3-x^2-y^2}} z\,\mathrm{d}z\right] \mathrm{d}(x,y) \\ &\quad +2\int_{1 \leqslant x^2+y^2 \leqslant 2} \left[\int_{\sqrt{x^2+y^2-1}}^{\sqrt{3-x^2-y^2}} z\,\mathrm{d}z\right] \mathrm{d}(x,y) \\ &= \int_{x^2+y^2 \leqslant 1} (3-x^2-y^2)\,\mathrm{d}(x,y) \\ &\quad + \int_{1 \leqslant x^2+y^2 \leqslant 2} (4-2x^2-2y^2)\,\mathrm{d}(x,y) \\ &= 2\pi \int_0^1 (3-r^2)\, r\,\mathrm{d}r \\ &\quad +2\pi \int_1^{\sqrt{2}} (4-2r^2)\, r\,\mathrm{d}r \\ &= \frac{5\pi}{2} + \pi = \frac{7\pi}{2}. \end{aligned} \tag{24}$$

Question 4. *Let $D \subset \mathbb{R}^2$. Let $T(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \end{pmatrix}$: $\mathbb{R}^2 \mapsto \mathbb{R}^2$ be a C^1 bijection such that $\det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right) \neq 0$ everywhere. Assume that $D, T^{-1}(D)$ both satisfy the hypothesis of Green's Theorem.*

Prove

$$\mu(D) = \int_{T^{-1}(D)} \left|\det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right)\right| \mathrm{d}(u,v) \tag{25}$$

using Green's Theorem.

Proof. Since $\det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right) \neq 0$ and is continuous, it is either >0 everywhere or <0 everywhere.

We take a parametrization $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, $t \in [a,b]$, of $\partial T^{-1}(D)$ such that the induced parametrization $\begin{pmatrix} X(u(t),v(t)) \\ Y(u(t),v(t)) \end{pmatrix}$ of ∂D satisfies the hypothesis of Green's Theorem.

Now calculate

$$\begin{aligned}
\mu(D) &= \left|\int_{\partial D} x\,\mathrm{d}y\right| \\
&= \left|\int_a^b X(u(t),v(t))\,\frac{\mathrm{d}}{\mathrm{d}t} Y(u(t),v(t))\,\mathrm{d}t\right| \\
&= \left|\int_a^b X Y_u u' + X Y_v v'\,\mathrm{d}t\right| \\
&= \left|\int_{\partial T^{-1}(D)} X Y_u\,\mathrm{d}u + X Y_v\,\mathrm{d}v\right| \\
&= \left|\int_{T^{-1}(D)} \frac{\partial(XY_v)}{\partial u} - \frac{\partial(XY_u)}{\partial v}\,\mathrm{d}(u,v)\right| \\
&= \left|\int_{T^{-1}(D)} [X_u Y_v - X_v Y_u]\,\mathrm{d}(u,v)\right| \\
&= \left|\int_{T^{-1}(D)} \det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right) \mathrm{d}(u,v)\right| \\
&= \int_{T^{-1}(D)} \left|\det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right)\right| \mathrm{d}(u,v).
\end{aligned} \tag{26}$$

The last equality holds because $\det\left(\frac{\partial(X,Y)}{\partial(u,v)}\right)$ is either >0 everywhere or <0 everywhere. ☐

Question 5. *Let S be a closed C^1 surface given by $\Phi = 0$ where $\Phi: \mathbb{R}^3 \mapsto \mathbb{R}$ is C^1 and satisfy $\operatorname{grad}\Phi \neq 0$ everywhere. Prove that the area of S is given by*

$$-\int_V \operatorname{div}\left(\frac{\operatorname{grad}\Phi}{\|\operatorname{grad}\Phi\|}\right) \mathrm{d}\boldsymbol{x} \tag{27}$$

where $V := \{\Phi > 0\}$ is the region enclosed by S.

Solution. Take any $(x_0, y_0, z_0) \in S$ and let $(x(t), y(t), z(t))$ be any C^1 curve on S passing (x_0, y_0, z_0). Then we have

$$\Phi(x(t), y(t), z(t)) = 0 \Longrightarrow \frac{\mathrm{d}\Phi}{\mathrm{d}t}(x_0, y_0, z_0) = 0. \tag{28}$$

Application of chain rule gives

$$(\operatorname{grad}\Phi)\cdot\begin{pmatrix} x'(t_0)\\ y'(t_0)\\ z'(t_0)\end{pmatrix}=0. \tag{29}$$

Therefore $\operatorname{grad}\Phi$ is normal to S. Then the outer normal is

$$\boldsymbol{n}=-\frac{\operatorname{grad}\Phi}{\|\operatorname{grad}\Phi\|}. \tag{30}$$

Application of Gauss's Theorem to yields

$$\text{Area}=\int_S \boldsymbol{n}\cdot\mathbf{d}\boldsymbol{S}=\int_S -\frac{\operatorname{grad}\Phi}{\|\operatorname{grad}\Phi\|}\cdot\mathbf{d}\boldsymbol{S}=-\int_V \operatorname{div}\left(\frac{\operatorname{grad}\Phi}{\|\operatorname{grad}\Phi\|}\right)\mathrm{d}\boldsymbol{x}. \tag{31}$$

Thus ends the proof.

Question 6. *Let $V\subset\mathbb{R}^3$ and let ∂V be C^1 oriented by outer normal $\boldsymbol{n}$. Let $u\colon\mathbb{R}^3\mapsto\mathbb{R}$, $\boldsymbol{f}\colon\mathbb{R}^3\mapsto\mathbb{R}^3$ be C^1. Prove that*

$$\int_{\partial V} u\,\boldsymbol{n}\,\mathrm{d}S=\int_V(\operatorname{grad}u)\,\mathrm{d}\boldsymbol{x};\qquad \int_{\partial V}\boldsymbol{n}\times\boldsymbol{f}\,\mathrm{d}S=\int_V(\operatorname{curl}\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}. \tag{32}$$

Here if $a\in\mathbb{R}$ and $\boldsymbol{b}=\begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix}\in\mathbb{R}^3$, $a\,\boldsymbol{b}$ is defined as $\begin{pmatrix} a\,b_1\\ a\,b_2\\ a\,b_3\end{pmatrix}$.

Solution. Denote

$$\boldsymbol{f}_1:=\begin{pmatrix} u\\ 0\\ 0\end{pmatrix},\qquad \boldsymbol{f}_2:=\begin{pmatrix} 0\\ u\\ 0\end{pmatrix},\qquad \boldsymbol{f}_3:=\begin{pmatrix} 0\\ 0\\ u\end{pmatrix}. \tag{33}$$

Then

$$\int_{\partial V} u\,\boldsymbol{n}\,\mathrm{d}S=\begin{pmatrix} \int_{\partial V}\boldsymbol{f}_1\cdot\boldsymbol{n}\,\mathrm{d}S\\ \int_{\partial V}\boldsymbol{f}_2\cdot\boldsymbol{n}\,\mathrm{d}S\\ \int_{\partial V}\boldsymbol{f}_2\cdot\boldsymbol{n}\,\mathrm{d}S\end{pmatrix}=\begin{pmatrix} \int_V\frac{\partial u}{\partial x}\,\mathrm{d}(x,y,z)\\ \int_V\frac{\partial u}{\partial y}\,\mathrm{d}(x,y,z)\\ \int_V\frac{\partial u}{\partial z}\,\mathrm{d}(x,y,z)\end{pmatrix}=\int_V(\operatorname{grad}u)\,\mathrm{d}\boldsymbol{x}. \tag{34}$$

Similarly, as

$$\boldsymbol{n}\times\boldsymbol{f}=\begin{pmatrix} n_2\,f_3-n_3\,f_2\\ n_3\,f_1-n_1\,f_3\\ n_1\,f_2-n_2\,f_1\end{pmatrix}=\begin{pmatrix} 0\cdot n_1+f_3\cdot n_2+(-f_2)\cdot n_3\\ (-f_3)\cdot n_1+0\cdot n_2+f_1\cdot n_3\\ f_2\cdot n_1+(-f_1)\cdot n_2+0\cdot n_3\end{pmatrix}, \tag{35}$$

the result follows from Gauss's Theorem. For example

$$\int_{\partial V} 0\cdot n_1+f_3\cdot n_2+(-f_2)\cdot n_3\,\mathrm{d}S=\int_V\operatorname{div}\begin{pmatrix} 0\\ f_3\\ -f_2\end{pmatrix}\mathrm{d}\boldsymbol{x}=\int_V[\partial_2 f_3-\partial_3 f_2]\,\mathrm{d}\boldsymbol{x} \tag{36}$$

which is the first component of $\int_V(\operatorname{curl}\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}$.