Math 317 Winter 2014 Homework 5 Solutions

DUE MAR. 12 2P

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Let $x \in (0,1)$. Recall that it has decimal representation $x = 0.a_1 a_2 a_3 \ldots$, $a_n \in \{0,1,2,3,4,5,6,7,8,9\}$ means*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}. \tag{1}$$

Prove that

- *If $x \neq \frac{m}{10^n}$ for any $m, n \in \mathbb{N}$, then it has a unique decimal representation;*
- *If $x = \frac{m}{10^n}$ for some $m, n \in \mathbb{N}$, then it has exactly two decimal representations.*

Question 2. *Prove the following through explicit construction of bijections.*

a) *$\mathbb{R} - \mathbb{Q} \sim \mathbb{R}$; You can assume that $\mathbb{Q}$ has already been listed as $\{r_1, r_2, \ldots\}$.*

b) *$\mathbb{R} - A \sim \mathbb{R}$, where A is the set of all algebraic numbers. You can assume that A has already been listed as $\{a_1, a_2, a_3, \ldots\}$.*

Question 3. *Recall that if $(X, \leqslant)$ is a partially ordered set, then $x_0 \in Y \subseteq X$ is said to be*

- *a **least element** of Y if and only if for every $y \in Y$, $x_0 \leqslant y$.*
- *a **minimal element** of Y if and only if there is no $y \in Y$ such that $y < x_0$.*

a) *Let $(X, \leqslant)$ be a partially ordered set. Assume that every non-empty subset of X has a least element. Prove that $\leqslant$ is in fact a well-ordering.*

b) *Does the conclusion in a) still hold if we instead assume that every non-empty subset has a minimal element?*

c) *Does the conclusion in a) still hold if we instead assume that every non-empty subset has a unique minimal element?*

d) *Find a partially ordered set $(X, \leqslant)$ such that it has a unique minimal element but no least element.*

Question 4. *We define an ordered pair (a, b) as $\{\{a\}, \{a, b\}\}$.*

a) *Prove that $(a, b) = (c, d) \Longleftrightarrow a = c, b = d$. (Note: the case $a = b$ needs to be discussed separately).*

b) *Review lecture note for Weeks 5 & 6 and the 217 lecture note on "Numbers". Explain why any positive real number x can be identified as a*

set of sets of sets of sets of finite ordinals.

Question 5.

a) *Re-order $\mathbb{N}$ to obtain the following ordinal numbers:*

$$\omega + 7; \qquad \omega \cdot 2; \qquad \omega \cdot \omega + 1 \tag{2}$$

You don't need to justify your answers.

b) *Find ordinal numbers* α, β, γ, *calculate* $(\alpha+\beta)\cdot\gamma$ *and* $\alpha\cdot\gamma+\beta\cdot\gamma$ *to show* $(\alpha+\beta)\cdot\gamma \neq \alpha\cdot\gamma+\beta\cdot\gamma$. *You don't need to justify your calculations.*

Question 6. *Let* $A_1 := \{E \subseteq \mathbb{R} | E \text{ is open}\}$; $A_2 := \{E \subseteq \mathbb{R} | E \text{ is closed}\}$; $A_3 := \{E \subseteq \mathbb{R} | E \text{ is Jordan measurable}\}$; $A_4 := \{E \subseteq \mathbb{R}^2 | E \text{ is Jordan measurable}\}$; $A_5 := \{f : \mathbb{R}^2 \mapsto \mathbb{R} | f \text{ is Riemann integrable}\}$. *Find the cardinalities of* $A_1 - A_5$. *(Hint:*[1]*)*

Question 7. (Extra 1 pt) *Study the wiki page http://en.wikipedia.org/wiki/JPEG about the JPEG format. Explain why discrete cosine transform (instead of discrete sine, or classical Fourier expansion/transform) are used in encoding every* 8×8 *block.*

1. Cantor set has the same cardinality as $\mathbb{R}$.