Math 317 Winter 2014 Homework 4

Due Feb. 26 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Calculate the Fourier expansion of the function* $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ *on* $[-\pi, \pi]$. *Then use the expansion to prove*

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \tag{1}$$

Question 2. *Let* $f(x)$ *be an even function, that is* $\forall x \in \mathbb{R}$, $f(x) = f(-x)$. *Prove that its Fourier expansion on* $[-L, L]$ *is given by*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} \mathrm{d}x. \tag{2}$$

Question 3. *Let* $f(x)$ *be odd and* $f(x) = 1 - \cos 2x$ *for* $x > 0$. *Expand* $f(x)$ *to its Fourier series on* $[-\pi, \pi]$.

Question 4. *Let* $f(x)$ *be integrable on* $[-\pi, \pi]$. *Assume that its Fourier expansion on* $[-\pi, \pi]$ *is*

$$\frac{0}{2} + \sum_{n=1}^{\infty} [0 \cdot \cos(n x) + 0 \cdot \sin(n x)]. \tag{3}$$

Let $x_0 \in (-\pi, \pi)$. *Prove that, if* $f(x)$ *is continuous at* x_0, *then* $f(x_0) = 0$. *(Hint: Consider for large* k

$$\int_{-\pi}^{\pi} f(x) [p(x)]^k \mathrm{d}x \tag{4}$$

with $p(x) = \varepsilon + \cos x$ *for appropriate* $\varepsilon > 0$.*)*

Question 5. *A sequence* $\{K_n\}$ *are called "good kernels" if and only if the following hold:*

- *All the* K_n*'s are even;*
- *For any* $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} K_n(x) \mathrm{d}x = 1$;
- *There is* $M > 0$ *such that for every* $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} |K_n(x)| \mathrm{d}x \leqslant M$;
- *For any* $\delta > 0$, $\lim_{n \to \infty} \int_{|x| > \delta} |K_n(x)| \mathrm{d}x = 0$.

Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ *be continuous and with period* 2π.

a) *Prove that* $f_n(x) := \int_{-\pi}^{\pi} K_n(x - t) f(t) \mathrm{d}t$ *converges to* $f(x)$ *uniformly.*

b) **(Extra 1 pt)** *Prove that the Dirichlet kernel is not "good".*

Question 6. *A set* $S \subseteq \mathbb{R}^N$ *is called "perfect" if and only if* $S = S' := \{x \in \mathbb{R}^N | \exists x_n \in S, x_n \neq x, \lim_{n \to \infty} x_n = x\}$. *Prove that perfect sets are uncountable.*

Question 7. (Extra 3 pts) *Consider two power series at* $x = 0$. *Let* $E := \{x \in \mathbb{R} | \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n < \infty\}$. *Find the weakest condition on* E *to guarantee* $a_n = b_n$ *for all* n. *Justify your answer using material* ***from 117 − 317 only***.

Question 8. (Extra 2 pts) *Prove that Peano's curve is continuous and onto from* $[0,1]$ *to* $[0,1]^2$.