Math 317 Winter 2014 Homework 4 Solutions

Due Feb. 26 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Calculate the Fourier expansion of the function* $f(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$ *on* $[-\pi,\pi]$. *Then use the expansion to prove*

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}. \tag{1}$$

Solution. We have

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos n x\,\mathrm{d}x=\frac{1}{\pi}\left[\int_0^{\pi}\cos(n x)\,\mathrm{d}x-\int_{-\pi}^{0}\cos(n x)\,\mathrm{d}x\right]=0; \tag{2}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(n x)\,\mathrm{d}x\\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin(n x)\,\mathrm{d}x-\int_{-\pi}^{0}\sin(n x)\,\mathrm{d}x\right]\\ &= \frac{2}{n\pi}[1-\cos(n\pi)]\\ &= \frac{2[1-(-1)^n]}{n\pi}. \end{aligned} \tag{3}$$

Thus we have

$$f(x)\sim\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2m-1)x}{2m-1}. \tag{4}$$

Now notice that $f(x)$ satisfies the Holder condition on $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$, therefore

$$1=f\left(\frac{\pi}{2}\right)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}. \tag{5}$$

The conclusion then follows.

Question 2. *Let* $f(x)$ *be an even function, that is* $\forall x\in\mathbb{R}$, $f(x)=f(-x)$. *Prove that its Fourier expansion on* $[-L,L]$ *is given by*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L},\qquad a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x. \tag{6}$$

Solution. We have

$$\begin{aligned} a_n &= \frac{1}{L}\left[\int_0^L f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x + \int_{-L}^0 f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x\right] \\ &= \frac{1}{L}\left[\int_0^L f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x - \int_{-L}^0 f(-x)\cos\frac{n\,\pi(-x)}{L}\,\mathrm{d}(-x)\right] \\ &= \frac{1}{L}\left[\int_0^L f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x + \int_0^L f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x\right] \\ &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\,\pi\,x}{L}\,\mathrm{d}x. \end{aligned} \tag{7}$$

Similarly we prove $b_n = 0$.

Question 3. *Let $f(x)$ be odd and $f(x) = 1 - \cos 2\,x$ for $x > 0$. Expand $f(x)$ to its Fourier series on $[-\pi, \pi]$.*

Solution. As $f(x)$ is odd, similar to the previous problem we have $a_n = 0$. Now we calculate b_n. We compute for $n = 1, 2, 3, \ldots$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi (1-\cos 2\,x)\sin(n\,x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi \sin(n\,x)\,\mathrm{d}x - \frac{2}{\pi}\int_0^\pi \sin(n\,x)\cos(2\,x)\,\mathrm{d}x \\ &= -\frac{2}{n\,\pi}\cos(n\,x)|_0^\pi - \frac{1}{\pi}\int [\sin(n+2)\,x + \sin(n-2)\,x]\,\mathrm{d}x \\ &= \frac{2}{n\,\pi}[1-(-1)^n] - \frac{1}{\pi}\int_0^\pi \sin(n+2)\,x\,\mathrm{d}x - \frac{1}{\pi}\int_0^\pi \sin(n-2)\,x\,\mathrm{d}x. \end{aligned} \tag{8}$$

We evaluate

$$\int_0^\pi \sin(n+2)\,x\,\mathrm{d}x = -\frac{1}{n+2}\cos(n+2)\,x|_0^\pi = \frac{1-(-1)^{n+2}}{n+2}. \tag{9}$$

For the last term, there are two cases.

- If $n = 2$, then $\sin(n-2)\,x = 0$ and
$$\int_0^\pi \sin(n-2)\,x\,\mathrm{d}x = 0. \tag{10}$$
- If $n \neq 2$, we compute
$$\int_0^\pi \sin(n-2)\,x\,\mathrm{d}x = -\frac{1}{n-2}\cos(n-2)\,x|_0^\pi = \frac{1-(-1)^{n-2}}{n-2}. \tag{11}$$

Putting everything together, we have

$$b_n = \begin{cases} \left(\frac{2}{n} - \frac{1}{n+2}\right)\frac{1-(-1)^n}{\pi} = 0 & n = 2 \\ \left(\frac{2}{n} - \frac{1}{n-2} - \frac{1}{n+2}\right)\frac{1-(-1)^n}{\pi} & n \neq 2 \end{cases}. \tag{12}$$

Question 4. *Let $f(x)$ be integrable on $[-\pi, \pi]$. Assume that its Fourier expansion on $[-\pi, \pi]$ is*

$$\frac{0}{2} + \sum_{n=1}^\infty [0\cdot\cos(n\,x) + 0\cdot\sin(n\,x)]. \tag{13}$$

Let $x_0 \in (-\pi, \pi)$. Prove that, if $f(x)$ is continuous at x_0, then $f(x_0) = 0$. (Hint: Consider for large k

$$\int_{-\pi}^{\pi} f(x)\,[p(x)]^k\,\mathrm{d}x \tag{14}$$

with $p(x) = \varepsilon + \cos x$ for appropriate $\varepsilon > 0$.)

Solution. Assume $f(x_0) \neq 0$. Wlog we consider $x_0 = 0$ and $m := f(0) > 0$. (When $x_0 \neq 0$ we can either define $F(x) := f(x + x_0)$ or use $p(x) = \varepsilon + \cos(x - x_0)$)

Since f is continuous at 0, there is $\delta_1 > 0$ such that $f(x) \geqslant \frac{m}{2}$ for all $|x| \leqslant \delta_1$, and $\delta_2 > \delta_1$ such that $f(x) \geqslant 0$ for all $|x| \leqslant \delta_2$. Now take $\varepsilon > 0$ such that there is $\varepsilon_0 > 0$ such that $p(x) := \varepsilon + \cos x$ satisfies

$$p(x) \begin{cases} \geqslant 1 + \varepsilon_0 & |x| \leqslant \delta_1 \\ \leqslant 1 - \varepsilon_0 & |x| \geqslant \delta_2 \\ > 0 & |x| \in (\delta_1, \delta_2) \end{cases}. \tag{15}$$

Now that for such p we have in fact,

$$|p(x)| \leqslant 1 - \varepsilon_0, \qquad |x| \geqslant \delta_2. \tag{16}$$

Now since the Fourier expansion of $f(x)$ is 0,

$$\int_{-\pi}^{\pi} f(x)\,[p(x)]^k\,\mathrm{d}x = 0 \tag{17}$$

for all $k \in \mathbb{N}$.

On the other hand, we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,[p(x)]^k\,\mathrm{d}x &= \int_{-\delta_1}^{\delta_1} f(x)\,[p(x)]^k\,\mathrm{d}x \\ &\quad + \int_{\delta_1 < |x| < \delta_2} f(x)\,[p(x)]^k\,\mathrm{d}x \\ &\quad + \int_{|x| \geqslant \delta_2} f(x)\,[p(x)]^k\,\mathrm{d}x \\ &=: A + B + C. \end{aligned} \tag{18}$$

Now by our choices of δ_1, δ_2, we have

$$A \geqslant 2\,\delta_1 \frac{m}{2}(1 + \varepsilon_0)^k, \qquad B \geqslant 0, \qquad |C| \leqslant 2\,\pi\,M\,(1 - \varepsilon_0)^k \tag{19}$$

where $M := \sup_{[-\pi,\pi]} |f(x)|$ is finite due to the integrability of f. Thus

$$\int_{-\pi}^{\pi} f(x)\,[p(x)]^k\,\mathrm{d}x \geqslant \delta_1\,m\,(1 + \varepsilon_0)^k - 2\,\pi\,M\,(1 - \varepsilon_0)^k. \tag{20}$$

Taking

$$k > \log_{[(1+\varepsilon_0)/(1-\varepsilon_0)]}\left(\frac{2\,\pi\,M}{\delta_1\,m}\right) \tag{21}$$

we have for this k,

$$\int_{-\pi}^{\pi} f(x)\,[p(x)]^k\,\mathrm{d}x > 0, \tag{22}$$

thus contradicting (17).

Remark. A slightly different (maybe a bit more transparent) proof is as follows.

Since $f(x)$ is continuous at 0, there is $\delta > 0$ such that $f(x) > 0$ for all $|x| < \delta$. Now take $\varepsilon = 1 - \cos\delta$. Furthermore take $\delta_1 < \delta$ such that

$$\cos\delta_1 = \frac{1 + \cos\delta}{2}. \tag{23}$$

Now consider $\int_{|x|<\delta_1}$, $\int_{\delta_1 \leqslant |x| < \delta}$, $\int_{|x| \geqslant \delta}$.

Question 5. *A sequence $\{K_n\}$ are called "good kernels" if and only if the following hold:*

- *All the K_n's are even;*
- *For any $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} K_n(x)\,\mathrm{d}x = 1$;*
- *There is $M > 0$ such that for every $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} |K_n(x)|\,\mathrm{d}x \leqslant M$;*
- *For any $\delta > 0$, $\lim_{n\to\infty} \int_{|x|>\delta} |K_n(x)|\,\mathrm{d}x = 0$.*

Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be continuous and with period 2π.

a) Prove that $f_n(x) := \int_{-\pi}^{\pi} K_n(x-t)\, f(t)\,\mathrm{d}t$ converges to $f(x)$ uniformly.

b) **(Extra 1 pt)** *Prove that the Dirichlet kernel is not "good".*

Solution.

a) Let $\varepsilon > 0$ be arbitrary. As $f(x)$ is continuous on $[-2\pi, 2\pi]$ there is $\delta > 0$ such that

$$\forall x, y \in [-2\pi, 2\pi], \qquad |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{2M}. \tag{24}$$

Now since $f(x)$ is periodic with period 2π, for any $x, y \in \mathbb{R}$, there are $x', y' \in [-2\pi, 2\pi]$ such that

$$f(x') = f(x), f(y') = f(y), \quad |x' - y'| \leqslant |x - y|. \tag{25}$$

Thus for the above δ we have

$$\forall x, y \in \mathbb{R}, \qquad |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{2M}. \tag{26}$$

As $f(x)$ is continuous and periodic, it is bounded. That is there is $M_1 > 0$ such that

$$\forall x \in \mathbb{R}, \qquad |f(x)| < M_1. \tag{27}$$

Next for the δ chosen above, since $\lim_{n\to\infty} \int_{|x|>\delta} |K_n(x)|\,\mathrm{d}x = 0$ there is $N \in \mathbb{N}$ such that

$$\forall n > N, \qquad \int_{|x|>\delta} |K_n(x)|\,\mathrm{d}x < \frac{\varepsilon}{4M_1}. \tag{28}$$

Now for any such n and any $x \in [-\pi, \pi]$, we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \int_{-\pi}^{\pi} K_n(x-t)\, f(t)\,\mathrm{d}t - \int_{-\pi}^{\pi} K_n(x-t)\, f(x)\,\mathrm{d}t \right| \\ &= \left| \int_{-\pi}^{\pi} K_n(x-t)\,[f(t) - f(x)]\,\mathrm{d}t \right| \\ &= \left| \int_{-\pi}^{\pi} K_n(u)\,[f(x-u) - f(x)]\,\mathrm{d}u \right| \\ &\leqslant \int_{-\pi}^{\pi} |K_n(u)|\,|f(x-u) - f(x)|\,\mathrm{d}u \\ &= \int_{-\delta}^{\delta} |K_n(u)|\,|f(x-u) - f(x)|\,\mathrm{d}u + \int_{|x|>\delta} |K_n(u)|\,|f(x-u) - f(x)|\,\mathrm{d}u \\ &\leqslant \frac{\varepsilon}{2M} \int_{-\pi}^{\pi} |K_n(u)|\,\mathrm{d}u + 2M_1 \int_{|x|>\delta} |K_n(u)|\,\mathrm{d}u \\ &< \frac{\varepsilon}{2M} \cdot M + 2M_1 \cdot \frac{\varepsilon}{4M_1} \\ &= \varepsilon. \end{aligned} \tag{29}$$

Thus ends the proof.

b) We check the conditions one by one.

- D_N is even;
- As $D_N(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N} \cos(n\,x)$ it is clear that $\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 1$;
- We have

$$D_N(x) = \frac{\sin\frac{2N+1}{2}x}{2\pi\sin\frac{x}{2}}. \tag{30}$$

Therefore we have

$$\begin{aligned}
\pi\int_{-\pi}^{\pi}|D_N(x)|\,\mathrm{d}x &= \int_0^{\pi}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x \\
&> \int_0^{2\pi/(2N+1)}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x + \int_{\frac{2\pi}{2N+1}}^{\frac{4\pi}{2N+1}}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x \\
&\quad + \cdots + \int_{\frac{2(N-1)\pi}{2N+1}}^{\frac{2N\pi}{2N+1}}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\sin\frac{x}{2}}\,\mathrm{d}x \\
&> \int_0^{\frac{2\pi}{2N+1}}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\frac{\pi}{2N+1}}\,\mathrm{d}x + \cdots + \int_{\frac{2(N-1)\pi}{2N+1}}^{\frac{2N\pi}{2N+1}}\frac{\left|\sin\frac{2N+1}{2}x\right|}{\frac{N\pi}{2N+1}}\,\mathrm{d}x \\
&= \frac{2N+1}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\int_0^{\frac{2\pi}{2N+1}}\left|\sin\frac{2N+1}{2}x\right|\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\int_0^{\pi}|\sin x|\,\mathrm{d}x \\
&= \frac{4}{\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right).
\end{aligned} \tag{31}$$

In the above we have used the fact that $\left|\sin\frac{2N+1}{2}x\right|$ is periodic with period $\frac{2\pi}{2N+1}$. Now it's clear that

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|D_N(x)|\,\mathrm{d}x = \infty. \tag{32}$$

- Similarly, we can prove that for any $\delta > 0$, $\lim_{n\to\infty}\int_{|x|>\delta}|D_N(x)|\,\mathrm{d}x = \infty$.

Question 6. *A set $S \subseteq \mathbb{R}^N$ is called "perfect" if and only if $S = S' := \{x \in \mathbb{R}^N |\, \exists x_n \in S, x_n \neq x, \lim_{n\to\infty} x_n = x\}$. Prove that perfect sets are uncountable.*

Solution. Assume $S = \{x_1, x_2, \ldots\}$ is countable. Since S is perfect, there is a compact interval I_0 such that $x_1 \in I_0^o$, and $I_0^o \cap S$ is infinite.

Now take $I_1 \subset I_0^o$ such that

- $I_1^o \cap S \neq \varnothing$;
- $x_1 \notin I_1$;
- $\operatorname{diam}(I_1) < \frac{\operatorname{diam}(I_0)}{2}$.

As S is perfect, $S \cap I_1^o$ is infinite. Now repeat the above process, we obtain a sequence of nested intervals $I_{n+1} \subset I_n^o$ such that

$$\forall n, \qquad x_n \notin I_n \Longrightarrow x_n \notin I_m \text{ whenever } m \geqslant n, \qquad I_n^o \cap S \neq \varnothing, \quad \operatorname{diam}(I_{n+1}) < \frac{\operatorname{diam}(I_n)}{2}. \tag{33}$$

By the nested interval theorem there is a unique point $x \in \mathbb{R}^N$ such that

$$x = \cap_{n=1}^{\infty} I_n. \tag{34}$$

Since $x_n \notin I_m$ whenever $m \geqslant n$, we see that $\forall n \in \mathbb{N}$, $x_n \notin \cap_{n=1}^{\infty} I_n$ which means $x \notin S$. However by construction of I_n we have $I_n^o \cap S \neq \varnothing$ for all n which means $x \in S' = S$. Contradiction.

Question 7. (Extra 3 pts) *Consider two power series at $x = 0$. Let $E := \{x \in \mathbb{R} | \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n < \infty\}$. Find the weakest condition on E to guarantee $a_n = b_n$ for all n. Justify your answer using material **from 117 – 317 only**.*

Solution. The weakest condition on E is E *has a limit point in* $(-R, R)$, where $R := \sup_{x \in E} |x|$. Note that as E has a limit point, it must contain infinitely many points and consequently $R > 0$.

- We see that both series have radius of convergence at least R. This means, if we set

$$A(x) := \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} b_n x^n, \tag{35}$$

then $A(x)$ is defined on $(-R, R)$, satisfying $A(x) = 0$ on E.

- Now by properties of power series, if $R > 0$, we have, for $x \in (-R, R)$,

$$A(x) = \sum_{n=0}^{\infty} (a_n - b_n) x^n \tag{36}$$

and furthermore

$$a_n - b_n = \frac{A^{(n)}(0)}{n!}. \tag{37}$$

Therefore all we need is to find the smallest E guaranteeing $A^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

- **Necessity.** Assume otherwise, then either E is finite or the only limit point(s) of E is R (or $-R$, or both – as by Bolzano-Weierstrass E must have at least one limit point).

  In the former case, assume $E = \{x_1, \ldots, x_m\}$. Then we set $A(x) = (x - x_1) \cdots (x - x_m)$.

  In the latter case, one counter-example is

$$A(x) = \sin\left(\frac{1}{(x-1)^2}\right) \exp\left(-\frac{1}{(x-1)^2}\right). \tag{38}$$

- **Sufficiency.** All we need to prove is the following:

  > Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$. Assume there are $x_n \in (-R, R)$ such that $f(x_n) = 0$ and $\lim_{n \to \infty} x_n = r \in (-R, R)$, then $a_n = 0$ for all $n = 0, 1, 2, \ldots$.

  Wlog $r > 0$. For simplicity of presentation we will assume $r < 1 < R$. Note that this can always be achieved through a change of variable $z = L\, x$ for appropriate $L > 0$. Also note that it suffices to prove $f(x) = 0$ for all x in some open interval containing 0.

  Since

$$\limsup_{n \to \infty} |a_n|^{1/n} = R^{-1} < 1, \tag{39}$$

there is $M>0$ such that

$$\forall n=0,1,2,\ldots. \quad |a_n|<M. \tag{40}$$

Now let $r_1\in(r,1)$ be arbitrary. For any $x\in(-r_1,r_1)$ we estimate

$$\left|f^{(k)}(x)\right|=\left|\sum_{n=k}^{\infty} n\,(n-1)\cdots(n-k+1)\,a_n\,x^{n-k}\right|\leqslant M\sum_{n=k}^{\infty} n\,(n-1)\cdots(n-k+1)r_1^{n-k}. \tag{41}$$

Notice that for any $y\in(0,1)$

$$\frac{1}{1-y}=\sum_{n=0}^{\infty} y^n. \tag{42}$$

By the theory of termwise differentiation we have

$$\left(\frac{1}{1-y}\right)^{(k)}=\sum_{n=k}^{\infty} n\,(n-1)\cdots(n-k+1)\,y^{n-k}. \tag{43}$$

Therefore

$$\sum_{n=k}^{\infty} n\,(n-1)\cdots(n-k+1)r_1=\frac{k!}{(1-r_1)^k}. \tag{44}$$

Consequently for any $x\in(-r_1,r_1)$

$$\frac{\left|f^{(k)}(x)\right|}{k!}\leqslant\frac{M}{(1-r_1)^k}. \tag{45}$$

Next take any $x_0\in(-r_1,r_1)$, we have, for any $x\in(-r_1,r_1)$,

$$\begin{aligned}\left|f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\,(x-x_0)^k\right| &= \left|\frac{f^{(k+1)}(c)}{(k+1)!}\,(x-x_0)^{k+1}\right|\\ &\leqslant \left|\frac{M}{(1-r_1)^{k+1}}\,(x-x_0)^{k+1}\right|.\end{aligned} \tag{46}$$

Therefore we have, for any x such that $|x-x_0|<1-r_1$,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}}{n!}\,(x-x_0)^n. \tag{47}$$

Now by similar argument as in the "weaker sufficient conditions" below, the fact that $f(x_n)=0$ and $\lim_{n\to\infty}x_n=r$ implies

$$f^{(n)}(r)=0 \tag{48}$$

for all $n=0,1,2,\ldots$. Taking $r_1=\frac{1+r}{2}$ and setting $x_0=r$ in the above analysis, we see that

$$f(x)=0 \tag{49}$$

for all $x\in(r-\delta(r),r+\delta(r))$ where $\delta(r)=\frac{1-r}{2}$.

Finally notice that once this is done, we can replace r by a smaller number, say $r-\frac{\delta(r)}{2}$ and repeat the above analysis, concluding that

$$f(x)=0 \tag{50}$$

for all $x \in (r - 2\,\delta(r), r + 2\,\delta(r))$. After finitely such iterations, we would have the desired open set containing 0 and on which $f(x)$ is identically 0. Since

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{51}$$

this means $a_n = 0$ for all n.

- **Weaker sufficient conditions with simpler proof:** $0 \in E$ *and is a limit point of* E. *That is there are* $x_n \in E$ *such that* $x_n \neq 0$, $\lim_{n\to\infty} x_n = 0$.

  Proof. By assumption $R > 0$. In this case at least one of $\{x_n > 0\}$ and $\{x_n < 0\}$ is infinite. Wlog we assume there are infinitely many $x_n > 0$. We can order them as

  $$x_1 > x_2 > \cdots > 0. \tag{52}$$

  Since $0 \in E$ we have

  $$a_0 - b_0 = A(0) = 0. \tag{53}$$

  Now since $A'(0)$ exists,

  $$A'(0) = \lim_{n\to\infty} \frac{A(x_n) - A(0)}{x_n - 0} = 0 \Longrightarrow a_1 = b_1. \tag{54}$$

  Next by MVT, there are

  $$\xi_n \in (x_{n+1}, x_n) \tag{55}$$

  such that $A'(\xi_n) = 0$. Then since $A''(0)$ exists we have

  $$A''(0) = \lim_{n\longrightarrow\infty} \frac{A'(\xi_n) - A'(0)}{\xi_n - 0} = 0 \Longrightarrow a_2 = b_2. \tag{56}$$

  Now it is easy to prove by induction that $a_n = b_n$ for all n. □

Question 8. (Extra 2 pts) *Prove that Peano's curve is continuous and onto from* $[0,1]$ *to* $[0,1]^2$.

Proof.

- Continuity. We see that

  $$\|f_{n+1}(x) - f_n(x)\| \leqslant \sqrt{2}\, 3^{-n} \tag{57}$$

  for every $n \in \mathbb{N}$ and $x \in [0,1]$. So by the M-test the convergence is uniform. Obviously each $f_n(x)$ is continuous. Therefore the limit $f(x)$ exists and is continuous.

- Onto. Consider any $(x,\ y) \in [0,\ 1]^2$. Then it belongs to at least one of the nine squares constructed in Step 2. Thus there is $t_2 \in [0,1]$ such that

  $$\|f_2(t_2) - (x,y)\| \leqslant \sqrt{2}\, 3^{-1}. \tag{58}$$

  Similarly we can get $t_3 \in [0,1]$ such that

  $$\|f_3(t_3) - (x,y)\| \leqslant \frac{\sqrt{2}}{3^2}, \qquad \ldots. \qquad \|f_n(t_n) - (x,y)\| \leqslant \frac{\sqrt{2}}{3^n}, \quad \ldots. \tag{59}$$

  Since $[0,1]$ is compact, by Bolzano-Weierstrass there is a subsequence

  $$t_{n_k} \longrightarrow t_\omega \in [0,1]. \tag{60}$$

  We will prove that $f(t_\omega) = (x,y)$.

Let $\varepsilon > 0$ be arbitrary. Since $f(t)$ is continuous there is $\delta > 0$ such that

$$|t - t_\omega| < \delta \Longrightarrow \| f(t) - f(t_\omega) \| < \frac{\varepsilon}{3}. \tag{61}$$

Set $K_1 \in \mathbb{N}$ be such that $k > K_1 \Longrightarrow |t_{n_k} - t_\omega| < \delta$.

On the other hand, as $f_n \longrightarrow f$ uniformly, there is $K_2 \in \mathbb{N}$ such that

$$k > K_2 \Longrightarrow \forall x \in [0,1], \quad \| f_{n_k}(t) - f(t) \| < \frac{\varepsilon}{3}. \tag{62}$$

Finally set K_3 such that

$$\frac{\sqrt{2}}{3^{n_{K_3}}} < \frac{\varepsilon}{3}. \tag{63}$$

Now take $K := \max\{K_1, K_2, K_3\}$. For every $k > K$, we have

$$\| f(t_\omega) - f_{n_k}(t_{n_k}) \| < \frac{2\,\varepsilon}{3} \Longrightarrow \| f(t_\omega) - (x, y) \| < \varepsilon. \tag{64}$$

Thus ends the proof. □