Math 317 Winter 2014 Homework 2

Due Jan. 29 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1.

a) *Prove the root test for* $\sum_{n=1}^{\infty} a_n$:

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1 \Longrightarrow \text{convergent}; \qquad \limsup_{n\to\infty} |a_n|^{1/n} > 1 \Longrightarrow \text{divergent}. \tag{1}$$

b) *Point out the mistake in my online lecture notes.*

Question 2. *Prove the following.*

a) $f_n(x) = \frac{n^2 x^2 - 3}{n^2 x + n x + 1}$ *converges uniformly on* $[2, 3]$;

b) $\sum_{n=1}^{\infty} x^3 e^{-n^2 x}$ *converges uniformly on* $(0, \infty)$.

Question 3.

a) *Prove* ***by definition*** *that if* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly on* $[a, b]$, *then* $\lim_{n\to\infty} u_n(x) = 0$ *uniformly;*

b) *Show that* $\lim_{n\to\infty} u_n(x) = 0$ *uniformly* $\not\Longrightarrow \sum_{n=1}^{\infty} u_n(x)$ *converges uniformly;*

c) ***Use part a)*** *to prove that* $\sum_{n=1}^{\infty} n e^{-nx}$ *converges on* $(0, \infty)$ *but not uniformly.*

Question 4. *Let* $u_n(x)$ *be Riemann integrable on* $[0, 1]$ *for all* n. *Assume that* $\sum_{n=1}^{\infty} u_n(x) = f(x)$ *uniformly on* $[0, 1]$. *Prove that* $f(x)$ *is also Riemann integrable on* $[0, 1]$ *and furthermore*

$$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\, \mathrm{d}x = \int_0^1 f(x)\, \mathrm{d}x. \tag{2}$$

Question 5. *Bernhard Riemann proposed* $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ *as a everywhere continuous but nowhere differentiable function on* $[0, 2\pi]$.

a) *Prove that* $f(x)$ *is continuous;*

b) *Calculate* $\int_0^{2\pi} f(x)\, \mathrm{d}x$. *Justify your answer;*

c) **(extra 3 pts)** *Comment on the differentiability of* $f(x)$. *Can you prove or disprove it? If not, why?*

Question 6. *Consider a function* $u(x, t)$ *defined on* $[0, 1] \times (0, \infty)$. *Assume that for each fixed* t_0, *the function* $u(x, t_0)$ *is continuous in* x.

a) *Give the definition for the convergence* $\lim_{t\to 0+} u(x, t) = f(x)$ *to be uniform on* $[0, 1]$.

b) *Prove that, if the convergence is uniform, then* $f(x)$ *is continuous.*

c) *Show through an example that when the convergence is not uniform,* $f(x)$ *may not be continuous.*