Math 317 Winter 2014 Homework 2 Solutions

Due Jan. 29 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1.

a) *Prove the root test for* $\sum_{n=1}^{\infty} a_n$*:*

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1 \Longrightarrow \text{convergent}; \qquad \limsup_{n\to\infty} |a_n|^{1/n} > 1 \Longrightarrow \text{divergent}. \tag{1}$$

b) *Point out the mistake in my online lecture notes.*

Solution.

Assume

$$\limsup_{n\to\infty} |a_n|^{1/n} = R < 1. \tag{2}$$

Then by definition of limsup, there is $N \in \mathbb{N}$ such that

$$\sup_{n>N} \left\{ |a_n|^{1/n} \right\} < r := \frac{R+1}{2}. \tag{3}$$

This means

$$\forall n > N, \qquad |a_n| < r^n \tag{4}$$

for $0 < r < 1$. Convergence now follows from comparison theorem.

On the other hand, if

$$\limsup_{n\to\infty} |a_n|^{1/n} = R > 1, \tag{5}$$

by definition

$$R = \limsup_{n\to\infty} |a_n|^{1/n} := \lim_{n\to\infty} \left[\sup_{k>n} |a_k|^{1/k} \right]. \tag{6}$$

Since $y_n := \sup_{k>n} |a_k|^{1/k}$ is decreasing, we have

$$\forall n \in \mathbb{N}, \qquad \sup_{k>n} |a_k|^{1/k} \geqslant R > 1 \Longrightarrow \sup_{k>n} |a_k| \geqslant R^k > 1 \Longrightarrow \limsup_{n-L\infty} |a_n| > 1. \tag{7}$$

Therefore $\lim_{n\to\infty} |a_n| = 0$ cannot hold (If $\lim_{n\to\infty} |a_n| = 0$ then $\limsup_{n\to\infty} |a_n| = 0$) and divergence follows.

Question 2. *Prove the following.*

a) $f_n(x) = \frac{n^2 x^2 - 3}{n^2 x + n x + 1}$ *converges uniformly on* $[2,3]$*;*

b) $\sum_{n=1}^{\infty} x^3 e^{-n^2 x}$ *converges uniformly on* $(0, \infty)$.

Solution.

a) First we have

$$\forall x \in [2,3], \qquad \lim_{n\to\infty} f_n(x) = x. \tag{8}$$

Now for any $\varepsilon > 0$, take $N > \frac{5}{\varepsilon}$, for every $n > N$, we have

$$\begin{aligned}\forall x \in [2,3], \qquad |f_n(x) - x| &= \left|\frac{(n^2 x^2 - 3) - x\,(n^2 x + n\,x + 1)}{n^2 x + n\,x + 1}\right| \\ &= \left|\frac{-3 - n\,x^2 - x}{n^2 x + n\,x + 1}\right| \\ &< \frac{3 + 9\,n + 3}{2\,n^2} \\ &< \frac{10\,n}{2\,n^2} \\ &= \frac{5}{n} < \frac{5}{N} < \varepsilon. \qquad (9)\end{aligned}$$

Thus the convergence is uniform.

b) For each $u_n(x) := x^3\, e^{-n^2 x}$, we have

$$u_n'(x) = 3\,x^2\, e^{-n^2 x} - n^2\, x^3\, e^{-n^2 x} = (3 - n^2\, x)\, x^2\, e^{-n^2 x} \qquad (10)$$

which is positive when $x < \frac{3}{n^2}$ and negative when $x > \frac{3}{n^2}$. Thus we have

$$\forall x \in (0, \infty), \qquad 0 < u_n(x) \leqslant u_n\left(\frac{3}{n^2}\right) < \frac{27}{n^6}. \qquad (11)$$

Now for any $\varepsilon > 0$ we take $N > \left(\frac{27}{\varepsilon}\right)^{1/6}$. Then for every $n > N$,

$$\forall x \in (0, \infty), \qquad |u_n(x) - 0| < \frac{27}{n^6} < \frac{27}{N^6} < \varepsilon. \qquad (12)$$

Therefore $u_n(x) \rightarrow 0$ uniformly on $(0, \infty)$.

Remark. The following fact, related to b), may be a bit curious:

Since x^3 is independent of n, we can write

$$\sum_{n=1}^{\infty} x^3\, e^{-n^2 x} = x^3 \left[\sum_{n=1}^{\infty} e^{-n^2 x}\right] \qquad (13)$$

and it suffices to prove the uniform convergence of $\sum_{n=1}^{\infty} e^{-n^2 x}$ on $(0, \infty)$. However this is clearly not true as $e^{-n^2 x}$ does not converge to 0 uniformly on $(0, \infty)$.

Please make sure you understand what is going on here.

Question 3.

a) *Prove* ***by definition*** *that if* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly on* $[a, b]$*, then* $\lim_{n\to\infty} u_n(x) = 0$ *uniformly;*

b) *Show that* $\lim_{n\to\infty} u_n(x) = 0$ *uniformly* $\not\Longrightarrow$ $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly;*

c) ***Use part a)*** *to prove that* $\sum_{n=1}^{\infty} n\, e^{-n x}$ *converges on* $(0, \infty)$ *but not uniformly.*

Solution.

a) Denote $u(x) := \sum_{n=1}^{\infty} u_n(x)$. Let $\varepsilon > 0$ be arbitrary.

Since $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\forall x \in [a, b], \qquad \left|\sum_{k=1}^{n} u_k(x) - u(x)\right| < \frac{\varepsilon}{2}. \qquad (14)$$

Now for such n, we have, by triangle inequality,

$$\forall x \in [a,b], \qquad |u_n(x)| \leqslant \left|\sum_{k=1}^{n+1} u_k(x) - u(x)\right| + \left|\sum_{k=1}^{n} u_k(x) - u(x)\right| < \varepsilon. \tag{15}$$

Thus by definition $u_n(x) \to 0$ uniformly.

b) A counter-example is $u_n(x) = \frac{1}{n}$ for all x.

c) First we show that it converges to 0. Let $x \in (0,\infty)$ be arbitrary. Then $x > 0$ and

$$\lim_{n\to\infty} n\, e^{-nx} = 0. \tag{16}$$

Denote $u_n(x) := n\, e^{-nx}$. We have

$$u_n\left(\frac{\ln n}{n}\right) = n\, e^{-\ln n} = 1. \tag{17}$$

Thus $u_n(x) \to 0$ cannot be uniform and the convergence of $\sum_{n=1}^{\infty} n\, e^{-nx}$ cannot be uniform.

Question 4. *Let $u_n(x)$ be Riemann integrable on $[0,1]$ for all n. Assume that $\sum_{n=1}^{\infty} u_n(x) = f(x)$ uniformly on $[0,1]$. Prove that $f(x)$ is also Riemann integrable on $[0,1]$ and furthermore*

$$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\, \mathrm{d}x = \int_0^1 f(x)\, \mathrm{d}x. \tag{18}$$

Solution. Let $\varepsilon > 0$ be arbitrary. Denote

$$f_n(x) = \sum_{k=1}^{n} u_k(x). \tag{19}$$

Since $f_n(x) \longrightarrow f(x)$ uniformly, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{4}. \tag{20}$$

As $f_{N+1}(x)$ is Riemann integrable, there is a partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ such that

$$U(f_{N+1}, P) - L(f_{N+1}, P) < \frac{\varepsilon}{2}. \tag{21}$$

Denote

$$M_i := \sup_{[x_i, x_{i+1}]} f_{N+1}, \qquad m_i := \inf_{[x_i, x_{i+1}]} f_{N+1}, \tag{22}$$

we have

$$\forall x \in [x_i, x_{i+1}], \qquad m_i \leqslant f_{N+1}(x) \leqslant M_i. \tag{23}$$

Together with (20) we have

$$m_i - \frac{\varepsilon}{4} \leqslant f(x) \leqslant M_i + \frac{\varepsilon}{4}. \tag{24}$$

Therefore

$$U(f,P) \leqslant \sum_{i=0}^{n-1} \left(M_i + \frac{\varepsilon}{4}\right)(x_{i+1} - x_i) = U(f_{N+1}, P) + \frac{\varepsilon}{4}. \tag{25}$$

Similarly

$$L(f,P) \geqslant L(f_{N+1}, P) - \frac{\varepsilon}{4}. \tag{26}$$

Thus we obtain

$$U(f,P) - L(f,P) < \varepsilon \tag{27}$$

and integrability of $f(x)$ follows.

Now (20) gives for all $n > N$,

$$\left| \int_0^1 f(x)\,\mathrm{d}x - \int_0^1 f_n(x)\,\mathrm{d}x \right| \leqslant \int_0^1 |f(x) - f_n(x)|\,\mathrm{d}x < \frac{\varepsilon}{4}. \tag{28}$$

This gives

$$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x. \tag{29}$$

Remark. Alternatively, one can use upper and lower integrals:

$$-\varepsilon < f - f_N < \varepsilon \Longrightarrow f_N - \varepsilon < f < f_N + \varepsilon \tag{30}$$

therefore

$$U(f) \leqslant U(f_N + \varepsilon) = U(f_N) + \varepsilon\,(b - a); \qquad L(f) \geqslant L(f_N - \varepsilon) = L(f_N) - \varepsilon\,(b - a). \tag{31}$$

However one should be careful as in general

$$U(f + g) \neq U(f) + U(g), \quad \text{etc.} \tag{32}$$

Question 5. *Bernhard Riemann proposed* $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ *as a everywhere continuous but nowhere differentiable function on* $[0, 2\pi]$*.*

a) Prove that $f(x)$ *is continuous;*

b) Calculate $\int_0^{2\pi} f(x)\,\mathrm{d}x$*. Justify your answer;*

c) **(extra 3 pts)** *Comment on the differentiability of* $f(x)$*. Can you prove or disprove it? If not, why?*

Solution.

a) Since on $\mathbb{R}$,

$$\left| \frac{\sin(n^2 x)}{n^2} \right| \leqslant \frac{1}{n^2}, \tag{33}$$

By Weierstrass' M-test $\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ converges uniformly and continuity immediately follows.

b) Since each $\frac{\sin(n^2 x)}{n^2}$ is continuous and thus integrable on $[0, 2\pi]$, uniform convergence gives

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\sin(n^2 x)}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty} 0 = 0. \tag{34}$$

c) There is no standard answer to this one.

Question 6. *Consider a function* $u(x, t)$ *defined on* $[0, 1] \times (0, \infty)$*. Assume that for each fixed* t_0*, the function* $u(x, t_0)$ *is continuous in* x*.*

a) Give the definition for the convergence $\lim_{t\to 0+} u(x, t) = f(x)$ *to be uniform on* $[0, 1]$*.*

b) Prove that, if the convergence is uniform, then $f(x)$ *is continuous.*

c) Show through an example that when the convergence is not uniform, $f(x)$ *may not be continuous.*

Solution.

a) $\forall \varepsilon > 0$, there is $\delta > 0$ such that for every $t \in (0, \delta)$,

$$\forall x \in [0, 1], \qquad |u(x, t) - f(x)| < \varepsilon. \tag{35}$$

b) Take any $x_0 \in [0, 1]$. We prove that $f(x)$ is continuous at x_0. Let $\varepsilon > 0$ be arbitrary. Then there is $t_0 > 0$ such that

$$\forall x \in [0, 1], \qquad |u(x, t_0) - f(x)| < \frac{\varepsilon}{3}. \tag{36}$$

Now since $u(x, t_0)$ is continuous at x_0, there is $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0 + \delta), \qquad |u(x, t_0) - u(x_0, t_0)| < \frac{\varepsilon}{3}. \tag{37}$$

Thus we have, for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - u(x, t_0) + u(x, t_0) - u(x_0, t_0) + u(x_0, t_0) - f(x_0)| \\ &\leqslant |f(x) - u(x, t_0)| + |u(x, t_0) - u(x_0, t_0)| + |u(x_0, t_0) - f(x_0)| \\ &< \varepsilon. \end{aligned} \tag{38}$$

Thus ends the proof.

c) Let $u(x, t) = e^{-x^2/t}$. Then

$$\lim_{t \to 0} u(x, t) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}. \tag{39}$$