# Math 317 Winter 2014 Homework 1 Solutions

Due Wednesday Jan. 15, 2014 2pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

**Question 1.** *Are the following series convergent or divergent? Justify your answers.*

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n!}}, \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}. \tag{1}$$

**Solution.**

- For the first series, apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{\sqrt{n+1}} \tag{2}$$

thus we have $\lim_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ which gives $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$. By ratio test we know the series converges.

- For the second series we notice

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n} = \sqrt{n+1}-\sqrt{n}. \tag{3}$$

Thus we have

$$s_n := \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n} \left(\sqrt{k+1}-\sqrt{k}\right) = \sqrt{n+1}-1. \tag{4}$$

Since $\lim_{n\longrightarrow\infty} s_n = \infty$, we have by definition of series convergence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \infty. \tag{5}$$

- Althernative method for the 2nd series. We prove:

$$\forall n \geqslant 1, \qquad \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{3\sqrt{n}}. \tag{6}$$

All we need to show is $\sqrt{n+1} < 2\sqrt{n} = \sqrt{4n}$ which immediately follows from $4n-(n+1) = 3n-1 > 0$ for all $n \geqslant 1$. Now since the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty, \tag{7}$$

so does $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$. Consequently $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

**Question 2.** *Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be non-negative series with $a_n > 0, b_n > 0$ for all $n \in \mathbb{N}$. Further assume that $\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$. Prove that $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.*

**Solution.** From the assumption we have (note that $a_1 > 0$ is used here)

$$\frac{a_2}{a_1} \leqslant \frac{b_2}{b_1} \Longrightarrow a_2 \leqslant \frac{a_1}{b_1} b_2; \tag{8}$$

$$\frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \leqslant \frac{b_3}{b_2} \cdot \frac{b_2}{b_1} \Longrightarrow \frac{a_3}{a_1} \leqslant \frac{b_3}{b_1} \Longrightarrow a_3 \leqslant \frac{a_1}{b_1} b_3; \tag{9}$$

In general we have

$$a_n \leqslant \frac{a_1}{b_1} b_n \tag{10}$$

for all $n \in \mathbb{N}$.

Now for any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} b_n$ converges, there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} b_k \right| < \frac{b_{N_0}}{a_{N_0}} \varepsilon. \tag{11}$$

Take $N = \max\{N_0, N_1\}$. We have for all $m > n > N$, (Note that we need the positivity of $a_k$ in the first inequality below)

$$\left| \sum_{k=n+1}^{m} a_k \right| \leqslant \left| \sum_{k=n+1}^{m} \frac{a_{N_0}}{b_{N_0}} b_k \right| = \frac{a_{N_0}}{b_{N_0}} \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon. \tag{12}$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges.

**Question 3.** *Prove by definition, without using improper integrals, that* $\sum_{n=1}^{\infty} \frac{1}{n \log_2 (n+1)} = \infty$.

**Proof.** We have

$$\frac{1}{1 \log_2 (1+1)} > \frac{1}{2}; \tag{13}$$

$$\frac{1}{2 \log_2 (2+1)} + \frac{1}{3 \log_2 (3+1)} > \frac{1}{4 \log_2 4} + \frac{1}{4 \log_2 4} = \frac{1}{4}; \tag{14}$$

$$\frac{1}{4 \log_2 (4+1)} + \cdots + \frac{1}{7 \log_2 (7+1)} > \frac{4}{8 \log_2 8} = \frac{1}{6}; \tag{15}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\frac{1}{2^{n-1} \log_2 (2^{n-1}+1)} + \frac{1}{(2^n - 1) \log_2 (2^n)} > \frac{2^{n-1}}{2^n n} = \frac{1}{2n}; \tag{16}$$

$$\vdots \quad \vdots \quad \vdots$$

Therefore

$$S_{2^n-1} := \sum_{k=1}^{2^n-1} \frac{1}{k \log_2 (k+1)} > \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k}. \tag{17}$$

Now for any $M > 0$, since $\left\{ \sum_{k=1}^{n} \frac{1}{k} \right\}$ is not bounded from above, there is $n_0 \in \mathbb{N}$ such that $\sum_{k=1}^{n_0} \frac{1}{k} > M$. This gives

$$S_{2^{n_0}-1} > M \tag{18}$$

and therefore $\{S_n\}$ is not bounded from above which means $\sum_{n=1}^{\infty} \frac{1}{n \log_2 (n+1)} = \infty$. □

**Question 4.** *Prove that if* $\sum_{n=1}^{\infty} a_n$ *is a convergent series of positive numbers, then so is* $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$ *(Note that this gives another proof of the fact that there can be not "largest" convergent series) (Hint:* [1]*)*

**Proof.** We prove that $a_n^{n/(n+1)} \leqslant 2\,a_n + 2^{-n}$. Note that this is equivalent to proving for any $a > 0$,

$$a \leqslant 2\,a^{1+1/n} + 2^{-n}. \tag{19}$$

We apply Young's inequality: $a, b > 0, 1/p + 1/q = 1$, then

$$a\,b \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{20}$$

We have

$$a = \left(2^{n/(n+1)}\,a\right) \cdot 2^{-n/(n+1)} \leqslant \frac{\left(2^{n/(n+1)}\,a\right)^{(n+1)/n}}{(n+1)/n} + \frac{\left(2^{-n/(n+1)}\right)^{n+1}}{n+1} \leqslant 2\,a^{1+1/n} + 2^{-n}. \tag{21}$$

Since both $\sum_{n=1}^{\infty} 2\,a_n$ and $\sum_{n=1}^{\infty} 2^{-n}$ converges, so does $\sum_{n=1}^{\infty} (2\,a_n + 2^{-n})$. The conclusion then follows from this fact together with that $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$ is non-negative. □

**Question 5.** *Let $a_n > 0$. Assume that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.*

**Proof.** There are two cases.

- $\{a_n\}$ is bounded. That is there is $M > 0$ such that $\forall n \in \mathbb{N}, 0 < a_n < M$. In this case we have

$$\frac{a_n}{1+a_n} > \frac{a_n}{1+M} \tag{22}$$

  and divergence follows from the divergence of $\sum_{n=1}^{\infty} \frac{a_n}{1+M}$.

- $\{a_n\}$ is not bounded. Thus for every $k \in \mathbb{N}$ that is $a_{n_k} > k$. This gives

$$\lim_{k\to\infty} \frac{a_{n_k}}{1+a_{n_k}} = 1. \tag{23}$$

  Thus $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$ does not hold and the series cannot converge. □

**Question 6.** *Assume $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ must also converge.*

**Proof.** We prove that the series is Cauchy.

Let $\varepsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, the partial sums $s_n := a_1 + \cdots + a_n$ are uniformly bounded in $n$, there is there is $M > 0$ such that for all $n \in \mathbb{N}$,

$$|s_n| < M. \tag{24}$$

Now take $N \in \mathbb{N}$ satisfying $N > \frac{3\,M}{\varepsilon}$. For any $m > n > N$ we calculate

$$\begin{aligned}
\frac{a_{n+1}}{n+1} + \cdots + \frac{a_m}{m} &= \frac{1}{n+1}(s_{n+1} - s_n) + \frac{1}{n+2}(s_{n+2} - s_{n+1}) + \cdots + \frac{1}{m}(s_m - s_{m-1}) \\
&= -\frac{s_n}{n+1} + s_{n+1}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + s_{m-1}\left(\frac{1}{m-1} - \frac{1}{m}\right) + \frac{s_m}{m} \\
&\leqslant \frac{M}{n+1} + M\left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right)\right] + \frac{M}{m} \\
&= \frac{M}{n+1} + M\left(\frac{1}{n+1} - \frac{1}{m}\right) + \frac{M}{m} \\
&< \frac{3\,M}{N} < \varepsilon.
\end{aligned} \tag{25}$$

1. *Apply Young's inequality to obtain $a_n^{n/(n+1)} \leqslant C_1\,a_n + C_2\,b_n$.*

Thus $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is Cauchy and therefore converges. $\square$