Comments on Homework 5

MARCH 13, 2014

1. Mistakes.

The following are popular mistakes. Try to fully understand why they are wrong.

i. Assume there are two or more unique representations... (Hint:[1])

ii. Consider the case $a \neq b$. Then

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \Longrightarrow \{a\} = \{c\}, \{a, b\} = \{c, d\}. \tag{1}$$

This is the only possibility because $\{a\}$ has one element, $\{c, d\}$ has two, so $\{a\} \neq \{c, d\}$. (Hint:[2])

iii. Re-order $\mathbb{N}$ as follows to obtain $\omega \cdot \omega + 1$:

$$2^1 < 2^2 < 2^3 < \cdots < 3^1 < 3^2 < 3^3 < \cdots < 5^1 < 5^2 < \cdots < 1 \tag{2}$$

(Hint:[3])

2. Exercises.

Some related exercises.

Exercise 1. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Prove that $\sum_{n=1}^{\infty} (a_n - b_n)$ is also convergent and furthermore

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \tag{3}$$

Note that this is used in all proofs of Question 1: $\sum_{n=1}^{\infty} \frac{a_n}{10^n} - \sum_{n=1}^{\infty} \frac{b_n}{10^n} = 0 \Longrightarrow \sum_{n=1}^{\infty} \frac{a_n - b_n}{10^n} = 0$. (Sol:[4])

Exercise 2. Find two infinite series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ such that

$$\left| \sum_{n=1}^{\infty} (a_n - b_n) \right| \geqslant |a_1 - b_1| \tag{6}$$

does not hold. (Hint:[5])

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1. "two or more unique" does not make sense.

2. $a \neq b$ does not automatically mean $c \neq d$.

3. A re-ordering should involve all elements in the set. But there are natural numbers not of the form p^k with p prime and $k \in \mathbb{N}$.

4. We prove $\sum_{n=1}^{\infty} (a_n - b_n)$ is Cauchy. Let $\varepsilon > 0$ be arbitrary. Then there are N_1, N_2 such that

$$\forall m > n > N_1, \qquad \left| \sum_{n+1}^{m} a_k \right| < \frac{\varepsilon}{2}; \qquad \forall m > n > N_2, \qquad \left| \sum_{n+1}^{m} b_k \right| < \frac{\varepsilon}{2}. \tag{4}$$

Take $N = \max\{N_1, N_2\}$. Then for all $m > n > N$, we have

$$\left| \sum_{n+1}^{m} (a_k - b_k) \right| \leqslant \left| \sum_{n+1}^{m} a_k \right| + \left| \sum_{n+1}^{m} b_k \right| < \varepsilon. \tag{5}$$

5. $1 + 0 + 0 + \cdots$ and $0 + 1 + 0 + \cdots$.

Exercise 3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Let $\{n_k\}, \{m_k\}$ be two subsequences of $\{n\}$ such that $\{n_1, n_2, ...\} \cup \{m_1, m_2, ...\} = \mathbb{N} = \{1, 2, ...\}$ and $\{n_1, n_2, ...\} \cap \{m_1, m_2, ...\} = \varnothing$. Now define a new series $\sum_{n=1}^{\infty} c_n$ as

$$c_n = \begin{cases} a_k & n = n_k \\ b_k & n = m_k \end{cases}. \tag{7}$$

Does it follow that $\sum_{n=1}^{\infty} c_n$ converges? If so what is the sum? (Hint:[6])

Exercise 4. Let X be a finite ordered set with a unique minimal element x_0. Prove that x_0 is in fact a least element. (Sol:[7])

Exercise 5. Let X be partially ordered and such that every finite non-empty subset has a least element. Prove that X is linearly ordered. Is X well-ordered? (Hint:[8])

Exercise 6. Let X be partially ordered and such that every countable non-empty subset has a least element. Is X well-ordered? (Hint:[9])

3. Other comments.

- When re-ordering a set, please give explicit rules, unless the rule is obvious.
- "Prove: There is a unique ..." means you need to both prove the existence and uniqueness.

6. Same idea as Exercise 1.

7. Assume the contrary. Then there is $x_1 \in X - \{x_0\}$ such that $x_0 < x_1$ does not hold. As x_0 is the unique minimal element, there must be $x_2 \in X$ such that $x_2 < x_1$. But $x_2 \neq x_0$ since otherwise $x_0 < x_1$. Therefore $x_2 \in X - \{x_0, x_1\}$. As x_2 is not minimal, there is $x_3 < x_2$. We claim that $x_3 \in X - \{x_0, x_1, x_2\}$. Obviously $x_3 \neq x_2$. If $x_3 = x_1$ then $x_1 < x_2$ contradicting $x_2 < x_1$; On the other hand if $x_3 = x_0$, we have $x_0 < x_2 < x_1 \Longrightarrow x_0 < x_1$ contradiction again. Therefore $x_3 \in X - \{x_0, x_1, x_2\}$. Now since x_3 is not minimal, we must have $x_4 \in X - \{x_0, x_1, x_2, x_3\}$ such that $x_4 < x_3$, and so on. As X is finite, this must stop at some finite step k, that is

$$x_k < x_{k-1} < \cdots < x_1; \quad \text{There is no } x \in X - \{x_0, x_1, ..., x_{k-1}\} \text{ such that } x < x_k \quad \text{and} \quad x_0 < x_k \text{ does not hold.} \tag{8}$$

Thus $x_k \neq x_0$ is a second minimal element. Contradiction.

8. Any $\{x, y\} \subseteq X$ has a least element $\Longrightarrow x \leqslant y$ or $x \geqslant y$; $\mathbb{Z}$.

9. By hypothesis every 2-element subset has a least element therefore X is in fact linearly ordered. Now assume there is a subset Y that does not have a least element. By hypothesis Y cannot be finite. Take $y_1 \in Y$, there is $y_2 \in Y - \{y_1\}$ such that $y_2 < y_1$. Then there is $y_3 < y_2 < y_1$, and so on.