Comments on Homework 3

February 6, 2014

1. Mistakes.

The following are popular mistakes. Try to fully understand why they are wrong.

i. Let $\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ be a power series. Then its radius of convergence can be calculated as

$$R^{-1} = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|. \tag{1}$$

(Make sure you understand why the radius of convergence can only be calculated from the root test.)

ii. Let $\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ be a power series with radius of convergence 1. Then the series converges uniformly on (x_0-1, x_0+1).

(Hint:[1] ; Make sure you have a counter-example.)

iii. Let $f(x)$ have Taylor expansion $\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ at $x=x_0$. Let the radius of convergence for $\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ be denoted R. If $R>0$, then we have

$$f(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^n \qquad \forall x \in (x_0-R, x_0+R). \tag{2}$$

(Hint:[2] This question can only be satisfactorily answered through complex analysis.)

iv. Let the infinite series of functions $\sum_{n=1}^{\infty} u_n(x)$ be such that

- Each $u_n(x)$ is integrable on $[a,b]$;
- $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$.

Then

$$\int \sum_{n=1}^{\infty} u_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \left[\int u_n(x)\,\mathrm{d}x\right]. \tag{3}$$

(Hint:[3] Make sure you understand the difference between $\int_a^b f(x)\mathrm{d}x$ and $\int f(x)\,\mathrm{d}x$.)

v. Let $R_1=0, R_2=\infty$. We define $R_1 R_2 = 0$.

vi. Assume that

$$f(x) = \sum_{n=1}^{\infty} u_n(x), \qquad \forall x \in (a,b). \tag{4}$$

1. Uniform convergence on $[-a,a]$ for every $0 \leqslant a < 1$. But not on $(-1,1)$.
2. $f(x) = \exp[-1/x^2]$ for $x \neq 0$ and $f(0)=0$.
3. The theorem is about definition integral.

Further assume that each $u_n(x)$ is continuous and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$. Then

$$f(a) = \sum_{n=1}^{\infty} u_n(a), \qquad f(b) = \sum_{n=1}^{\infty} u_n(b). \tag{5}$$

(Hint:[4])

2. Exercises.

Some related exercises.

Exercise 1. Find a power series such that its radius of convergence R does not equal $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. (Hint:[5])

Exercise 2. Let $a_n, b_n > 0$. Prove that

$$\limsup_{n\to\infty} (a_n + b_n)^{1/n} = \max\left(\limsup_{n\to\infty} a_n^{1/n}, \limsup_{n\to\infty} b_n^{1/n}\right). \tag{6}$$

How should we change the formula if we drop the conditions $a_n, b_n > 0$? (Hint:[6])

Exercise 3. Let $E_1, E_2, \ldots, E_m \subseteq \mathbb{R}$. Assume that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on each E_i, $i = 1, 2, 3, \ldots, m$. Prove that the series converges uniformly on $\cup_{k=1}^{m} E_k$. What if we allow m to be infinite?

Exercise 4. Let $\sum_{n=1}^{\infty} u_n(x)\, v_n(x)$ satisfy that

- There is $M > 0$ such that $\left|\sum_{k=1}^{n} u_k(x)\right| < M$ for all $n \in \mathbb{N}$ and all $x \in [a,b]$;
- There is a positive decreasing sequence $\{M_n\}$ such that $|v_n(x)| < M_n$ for all $n \in \mathbb{N}$ and all $x \in [a,b]$, and furthermore $\lim_{n\to\infty} M_n = 0$.

Does it follow that $\sum_{n=1}^{\infty} u_n(x)\, v_n(x)$ converge uniformly on $[a,b]$? Justify.

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4. $f(x)$ may not be continuous at a, b. Optional: Try to understand why a counter-example is hard to find.
5. $a_n = 2 + (-1)^n$.
6. $\max(a_n, b_n) < a_n + b_n < 2\max(a_n, b_n)$.