MATH 314 FALL 2012 MIDTERM

OCT. 23, 2012 2PM - 3:20PM. TOTAL 60 PTS

NAME: ID#:

- Please write clearly and show enough work.

Problem 1. (5 pts) A function $f(x): E \mapsto \mathbb{R}$ is said to be Lipschitz continuous if there is $M \in \mathbb{R}$ such that for every $x, y \in E$, $|f(x) - f(y)| \leqslant M\ |x - y|$. Write down the logical statement of "$f(x)$ is not Lipschitz continuous".

Problem 2. (5 pts) Let $f(x): X \mapsto Y$ satisfy: For any $A, B \subseteq X$, if $A \cap B = \varnothing$ then $f(A) \cap f(B) = \varnothing$. Prove that f is one-to-one.

Problem 3. (10 pts) Find the following limits. Justify your answers. (You can use the convergence/divergence of $x_n = n^a$ without proof)

a) **(3 pts)** $\lim_{n\longrightarrow\infty}\left[\sqrt{n^2+4\,n}-\sqrt{n^2-2\,n}\,\right]$.

b) **(3 pts)** $\lim_{x\longrightarrow 1}\frac{x^2-1}{x^3-1}$.

c) **(4 pts)** $\lim_{n\longrightarrow\infty}\frac{5\,x_n}{n^3+2\,n+1}$ where x_n satisfies $|x_n|\leqslant 3\,n$ for all $n\in\mathbb{N}$.

Problem 4. (10 pts) Let $A = \{x \in \mathbb{R}: e^{x^2} > e\}$, $B = \{x \in \mathbb{R}: x > 0, \ln x \leqslant 0\}$.

a) **(4 pts)** Express $A, B, A \cap B, A \cup B$ using intervals.

b) **(6 pts)** Among the four sets above, which is/are open? Which is/are closed? Justify your answers.

Problem 5. (10 pts) Let $x_n = (-1)^n - e^{-n}$ and $E = \{x_n : n \in \mathbb{N}\}$. ($\mathbb{N} = \{1, 2, 3, ...\}$)

a) **(6 pts)** Find $\max E, \sup E, \min E, \inf E$. Justify your answers.

b) **(4 pts)** Calculate $\limsup_{n \longrightarrow \infty} x_n$ and $\liminf_{n \longrightarrow \infty} x_n$.

Problem 6. (10 pts) Let $x_0 = 25$ and define x_n through

$$x_{n+1} = \frac{3\,x_n}{7} - 8. \tag{1}$$

Prove that $\{x_n\}$ converges and find its limit. (You can use the formula $1 + r + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$ without proof)

Problem 7. (5 pts) Is $f(x)=\begin{cases} \dfrac{(\cos x)(\sin x^2)}{x^2} & x\neq 0 \\ 1 & x=0 \end{cases}$ continuous for all $x\in\mathbb{R}$? Justify your answer. (You can use $\lim_{x\longrightarrow 0}\frac{\sin x}{x}=1$ without proof).

Problem 8. (5 pts) Let $f: \mathbb{R} \mapsto \mathbb{R}$, $g: \mathbb{R} \mapsto \mathbb{R}$ be continuous functions. Assume $f(x) > 0$ for all $x \in \mathbb{R}$.

a) **(4 pts)** Prove that for any closed interval $[a, b]$ with $a, b \in \mathbb{R}$, there is $\delta_0 > 0$ such that for all $0 \leqslant \delta < \delta_0$, $f(x) + \delta g(x) > 0$ for all $x \in [a, b]$.

b) **(1 pt)** Is the claim still true when $a = -\infty$ or $b = \infty$ (or both)?