Math 314 Fall 2012 Final Exam

Dec. 10, 2012 2pm - 4pm, MEC 2-3

NAME: ID#:

- There are 11 Problems, total 100 points.
- Please **justify** all your answers:
  - For calculation problems, "justify" means show enough steps.
- Last 3 pages (pages 13 – 15) are blank. Detach one or more if you need scrap paper.

GOOD LUCK!!

Problem 1. (10 pts) Let $f(x)=\begin{cases} \dfrac{1-\cos x}{e^x-1-x} & x\neq 0 \\ c & x=0 \end{cases}$ for some $c\in\mathbb{R}$.

a) **(4 pts)** Prove that the only solution to $e^x-1-x=0$ is $x=0$.

b) **(6 pts)** For what value of c is $f(x)$ continuous at all $x\in\mathbb{R}$?

Problem 2. (10 pts) Let $f(x)\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mapsto \mathbb{R}$ be defined by $f(x) = 2\tan x - x$.

a) **(5 pts)** Prove that $f(x)$ is strictly increasing.

b) **(5 pts)** Let g be the inverse function of f. Calculate $g'(0)$.

Problem 3. (10 pts) Let $f(x) = \exp[\cos x]$.

a) **(6 pts)** Calculate $f'(x)$, $f''(x)$, $f'''(x)$.

b) **(4 pts)** Obtain Taylor polynomial to degree 2 with Lagrange form of remainder at $x_0 = 0$.

Problem 4. (10 pts)

a) **(5 pts)** Calculate $\int_0^1 x^{-1/3} e^{-x^{1/3}}\,\mathrm{d}x$.

b) **(5 pts)** Prove that $f(x)=\dfrac{1}{(1+x)^3}$ is improperly integrable on $(0,\infty)$ and find $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(1+x)^3}$.

Problem 5. (10 pts) Prove that $\sum_{n=1}^{\infty} n^2 x^n$ converges when $|x| < 1$ and diverges when $|x| \geqslant 1$.

Problem 6. (10 pts) Let $a > 1$. Prove that $\lim_{n \longrightarrow \infty} [(n+4)^a - n^a] = \infty$. You can use $(x^a)' = a\, x^{a-1}$ and the limit of x^a without justification.

Problem 7. (10 pts) Let $g(x)$ be integrable on $[a, b]$ and define $F(x) = \int_a^x g(t)\,\mathrm{d}t$.

a) **(5 pts)** Prove that if $g(x) > 0$ and is continuous on $[a, b]$, then $\exists \delta > 0$ such that $F'(x) \geqslant \delta$ for all $x \in (a, b)$.

b) **(5 pts)** Prove that if $g(x)$ is increasing on (a, b) and is **not** continuous at $x_0 \in (a, b)$, then $F(x)$ is **not** differentiable at x_0.

Problem 8. (10 pts) Let $\varphi(x) = \int_0^x \ln(\cos t)\,\mathrm{d}t$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

a) **(5 pts)** Prove that $\varphi(x) = x\ln 2 + 2\,\varphi\!\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2\,\varphi\!\left(\frac{\pi}{4} - \frac{x}{2}\right)$.

b) **(5 pts)** Calculate $\int_0^{\pi/2} \ln(\cos t)\,\mathrm{d}t$. Note that you can still solve b) even if you don't know how to prove a).

Problem 9. (10 pts) Let $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverge. Prove

a) **(5 pts)** $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges.

b) **(5 pts)** $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges.

Problem 10. (5 pts) Let $f(x)\colon \mathbb{R} \mapsto \mathbb{R}$ be a bounded function, that is there is $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in \mathbb{R}$. Prove that if $f''(x) \geqslant 0$ for all $x \in \mathbb{R}$ then f is constant.

Problem 11. (5 pts) Let $f(x): [a, b] \mapsto \mathbb{R}$ with $a, b \in \mathbb{R}$. For any partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$, we define $V_a^b(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ and then $V_a^b(f) := \sup_P V_a^b(f, P)$. If $V_a^b(f)$ is finite we say $f(x)$ is BV.

a) **(2 pt)** Prove that if $f(x)$ is BV then it is integrable.

b) **(1 pt)** Give an example of a BV function that is not continuous.

c) **(1 pt)** Assume that $f'(x)$ exists and $|f'(x)|$ is integrable on $[a, b]$. Prove that $V_a^b(f) \leqslant \int_a^b |f'(x)| \, \mathrm{d}x$.

d) **(1 pt)** Prove that with the assumptions in c), we in fact have $V_a^b(f) = \int_a^b |f'(x)| \, \mathrm{d}x$.