Math 314 Fall 2013 Homework 9 Solutions

Due Wednesday Nov. 20 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $a>0$. Use Mean Value Theorem to prove*

$$\sqrt{2+a}-\sqrt{1+a}<\sqrt{1+a}-\sqrt{a}. \tag{1}$$

You can use $(x^a)'=a\,x^{a-1}$ without proof.

Solution. Consider the function $f(x)=\sqrt{x}$. By mean value theorem we have

$$f(x)-f(y)=f'(\xi)\,(x-y)=\frac{1}{2\sqrt{\xi}}\,(x-y). \tag{2}$$

Now setting x,y to be $2+a,1+a$, and $1+a,a$ respectively, we have

$$\sqrt{2+a}-\sqrt{1+a}=\frac{1}{2\sqrt{\xi_1}}; \qquad \sqrt{1+a}-\sqrt{a}=\frac{1}{2\sqrt{\xi_2}} \tag{3}$$

with

$$\xi_1\in(2+a,1+a), \qquad \xi_2\in(1+a,a). \tag{4}$$

Thus $\xi_1>\xi_2$ and consequently

$$\sqrt{2+a}-\sqrt{1+a}<\sqrt{1+a}-\sqrt{a}. \tag{5}$$

Question 2. *In the proof of L'Hospital's rule, we arrive at: For every $x\neq x_0$, there is c between x, x_0 with $c\neq x_0,x$, such that*

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}. \tag{6}$$

Assume that

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L\in\mathbb{R}. \tag{7}$$

Prove by definition of limit that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=L. \tag{8}$$

Proof. For any $\varepsilon>0$, since $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L$ there is $\delta_1>0$ such that for all $0<|x-x_0|<\delta_1$,

$$\left|\frac{f'(x)}{g'(x)}-L\right|<\varepsilon. \tag{9}$$

Now set $\delta=\delta_1$. For any $x\neq x_0$ such that $0<|x-x_0|<\delta$, since c is between x_0 and x and furthermore $c\neq x_0, c\neq x$, we have

$$0<|c-x_0|<|x-x_0|<\delta=\delta_1. \tag{10}$$

Thus $0<|c-x_0|<\delta_1$ and consequently

$$\left|\frac{f(x)}{g(x)}-L\right|=\left|\frac{f'(c)}{g'(c)}-L\right|<\varepsilon. \tag{11}$$

By definition $\lim_{x\to x_0}\frac{f(x)}{g(x)}=L$. □

Question 3. *Calculate*

$$\lim_{x\to 0}\frac{\sin x}{x\cos x} \tag{12}$$

using L'Hospital's rule. You should explicitly check that the four conditions for the application of the rule are satisfied. In particular, make your (a,b) explicit.

Solution. Set $f(x)=\sin x$, $g(x)=x\cos x$. $(a,b)=\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$. $x_0=0$.

1. $f(x), g(x)$ are differentiable on $(a,b)-\{x_0\}$.
   Since $x, \cos x$ are differentiable at every x, so is their product $x\cos x$. Furthermore $\sin x$ is differentiable at every x. Therefore this condition is satisfied.
2. $\lim_{x\longrightarrow x_0} f(x)=\lim_{x\longrightarrow x_0} g(x)=0$.
   Since $x\cos x$ and $\sin x$ are continuous at 0, we have
   $$\lim_{x\to 0}(x\cos x)=0\cdot\cos 0=0; \qquad \lim_{x\to 0}\sin x=\sin 0=0. \tag{13}$$
3. $\lim_{x\longrightarrow x_0}\frac{f'(x)}{g'(x)}$ exists.
   Since $\lim_{x\to 0}[\cos x-x\sin x]=1\neq 0$, we have
   $$\lim_{x\to 0}\frac{\cos x}{\cos x-x\sin x}=\frac{1}{1-0}=1. \tag{14}$$
4. $g'(x)\neq 0$ for $x\in(a,b)-\{x_0\}$.
   For $x\in\left(-\frac{\pi}{6},\frac{\pi}{6}\right)-\{0\}$, we have
   $$\cos x>\frac{\sqrt{3}}{2}>\frac{1}{2}, \qquad |\sin x|<\frac{1}{2}\Longrightarrow |x\cos x|<\frac{\pi}{12}<\frac{1}{2}. \tag{15}$$
   Therefore
   $$\cos x-x\sin x>0 \tag{16}$$
   for all $x\in\left(-\frac{\pi}{6},\frac{\pi}{6}\right)-\{0\}$.

Thus L'Hospital's rule gives

$$\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\cos x}{\cos x-x\sin x}=1. \tag{17}$$

Question 4. *Calculate*

$$\lim_{x\longrightarrow 0}\frac{e^x-e^{-x}-2x}{x-\sin x} \tag{18}$$

using L'Hospital's rule. (Note for this problem you do not need to check the conditions explicitly)

Solution. We have

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{e^x-e^{-x}-2x}{x-\sin x} &= \lim_{x\longrightarrow 0}\frac{e^x+e^{-x}-2}{1-\cos x}\\ &= \lim_{x\longrightarrow 0}\frac{e^x-e^{-x}}{\sin x}\\ &= \lim_{x\longrightarrow 0}\frac{e^x+e^{-x}}{\cos x}=2.\end{aligned} \tag{19}$$

Question 5. *Prove the following "Naive L'Hospital's rule": Let $x_0 \in (a, b) \subseteq \mathbb{R}$. Let f, g be defined on (a, b) and satisfy*

1. $f(x_0) = g(x_0) = 0$;
2. f, g *are differentiable at* x_0;
3. $g'(x_0) \neq 0$.

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{20}$$

Solution.

Since f, g are differentiable at x_0 we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \qquad \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0). \tag{21}$$

Since $g'(x_0) \neq 0$ we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x) - f(x_0)}{x - x_0}}{\frac{g(x) - g(x_0)}{x - x_0}} = \frac{f'(x_0)}{g'(x_0)}. \tag{22}$$

Finally using the fact that $f(x_0) = g(x_0) = 0$ we obtain

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} \tag{23}$$

as desired.

Question 6. *Let* $f(x) = \sin 2x$.

a) *Calculate its Taylor expansion to degree 3 at $x_0 = 0$ with Lagrange form of remainder;*

b) *Let $P_3(x)$ be the Taylor polynomial obtained above. Prove that $|\sin 2x - P_3(x)| < \frac{1}{120}$ for all $-\frac{1}{2} < x < \frac{1}{2}$.*

Solution.

a) We have $f'(x) = 2\cos 2x$, $f'' = -4\sin 2x$, $f''' = -8\cos 2x$, $f^{(4)}(x) = 16\sin 2x$. Therefore the Taylor polynomial of degree 3 at $x_0 = 0$ with Lagrange remainder is

$$f(x) = 2x - \frac{4}{3}x^3 + \frac{2\sin(2\xi)}{3}x^4 \tag{24}$$

where ξ lies between 0 and x.

b) We have $P_3(x) = 2x - \frac{4}{3}x^3$. Noticing that $f^{(4)}(0) = 0$, we have in fact $P_3(x) = P_4(x)$ and

$$f(x) = 2x - \frac{4}{3}x^3 + \frac{4\cos(2\xi)}{15}x^5. \tag{25}$$

Therefore

$$|\sin 2x - P_3(x)| \leqslant \frac{4}{15}|x|^5 < \frac{1}{120}. \tag{26}$$