Math 314 Fall 2013 Homework 9

Due Wednesday Nov. 20 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $a>0$. Use Mean Value Theorem to prove*

$$\sqrt{2+a}-\sqrt{1+a}<\sqrt{1+a}-\sqrt{a}. \tag{1}$$

You can use $(x^a)'=a\,x^{a-1}$ without proof.

Question 2. *In the proof of L'Hospital's rule, we arrive at: For every $x\neq x_0$, there is c between x, x_0 with $c\neq x_0, x$, such that*

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}. \tag{2}$$

Assume that

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L\in\mathbb{R}. \tag{3}$$

Prove by definition of limit that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=L. \tag{4}$$

Question 3. *Calculate*

$$\lim_{x\to 0}\frac{\sin x}{x\cos x} \tag{5}$$

using L'Hospital's rule. You should explicitly check that the four conditions for the application of the rule are satisfied. In particular, make your (a,b) explicit.

Question 4. *Calculate*

$$\lim_{x\longrightarrow 0}\frac{e^x-e^{-x}-2\,x}{x-\sin x} \tag{6}$$

using L'Hospital's rule. (Note for this problem you do not need to check the conditions explicitly)

Question 5. *Prove the following "Naive L'Hospital's rule": Let $x_0\in(a,b)\subseteq\mathbb{R}$. Let f,g be defined on (a,b) and satisfy*

1. *$f(x_0)=g(x_0)=0$;*
2. *f,g are differentiable at x_0;*
3. *$g'(x_0)\neq 0$.*

Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f'(x_0)}{g'(x_0)}. \tag{7}$$

Question 6. *Let $f(x)=\sin 2\,x$.*

a) *Calculate its Taylor expansion to degree 3 at $x_0=0$ with Lagrange form of remainder;*

b) *Let $P_3(x)$ be the Taylor polynomial obtained above. Prove that $|\sin 2x-P_3(x)|<\frac{1}{120}$ for all $-\frac{1}{2}<x<\frac{1}{2}$.*