Math 314 Fall 2013 Homework 8 Solutions

Due Wednesday Nov. 13 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $f(x) = \exp[x \ln x]$. Calculate $f'(x)$.*

Solution. We have

$$\begin{aligned} f'(x) &= \exp[x \ln x]\,(x \ln x)' \\ &= \exp[x \ln x]\,(x' \ln x + x\,(\ln x)') \\ &= \exp[x \ln x]\,(\ln x + 1). \end{aligned} \tag{1}$$

Question 2. *Let $f(x) = \arccos x$. Calculate $f'(x)$.*

Solution. We have

$$f'(x) = \frac{1}{(\cos y)'} = -\frac{1}{\sin y}. \tag{2}$$

Here $y = \arccos x \in [0, \pi)$. Therefore $\sin y \geqslant 0$ and

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}. \tag{3}$$

So

$$f_3'(x) = -\frac{1}{\sqrt{1 - x^2}}. \tag{4}$$

Question 3. *Is $x_0 = \frac{1}{20\,\pi}$ a local maximizer for $f(x) = (1 + (\sin x)^4) \cos\left(\frac{1}{x}\right)$? Justify your answer.*

Solution. We calculate

$$f'(x) = 4\,(\sin x)^3 \cos x \cos\left(\frac{1}{x}\right) + \frac{1 + (\sin x)^4}{x^2} \sin\left(\frac{1}{x}\right). \tag{5}$$

This gives

$$f'(x_0) = 4\left(\sin\left(\frac{1}{20\,\pi}\right)\right)^3 \cos\left(\frac{1}{20\,\pi}\right) \neq 0 \tag{6}$$

so x_0 cannot be a local maximizer for $f(x)$.

Question 4. *Prove Cauchy's Mean Value Theorem.*

Solution. Take $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}\,g(x)$. Since f, g are continuous on $[a, b]$ and differentiable on (a, b), so is $h(x)$. Apply Mean Value Theorem to h we have $\xi \in (a, b)$ such that

$$h'(\xi) = \frac{h(b) - h(a)}{b - a}. \tag{7}$$

Since

$$h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)}\,g'(\xi) \tag{8}$$

and

$$h(b) - h(a) = 0 \tag{9}$$

we see that

$$f'(\xi) = \frac{f(b) - f(a)}{g(b) - g(a)}\,g'(\xi) \Longrightarrow \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{10}$$

Question 5. *Let $f(x) = e^x - 1 - \sin x$. Prove that $f(x) \geqslant 0$ for all $x \geqslant 0$.*

Solution. Since $f(0) = 0$ it suffices to prove $f(x)$ is increasing.

As f is differentiable, we calculate

$$f'(x) = e^x - \cos x \geqslant 0 \tag{11}$$

for all $x > 0$. Therefore f is increasing and we have

$$\forall x \geqslant 0, \qquad f(x) \geqslant f(0) = 0. \tag{12}$$

Question 6. *Prove*

$$\forall x \in (-1/2, 1/2), \qquad 3 \arccos x - \arccos (3\,x - 4\,x^3) = \pi. \tag{13}$$

You can use the result from Question 2.

Solution. Let $h(x) = 3 \arccos x - \arccos (3\,x - 4\,x^3)$. Then we show

- $h'(x) = 0$.

$$\begin{aligned} h'(x) &= -\frac{3}{\sqrt{1-x^2}} + \frac{3 - 12\,x^2}{\sqrt{1-(3\,x-4\,x^3)^2}} \\ &= -\frac{3}{\sqrt{1-x^2}} + \frac{3\,(1-4\,x^2)}{\sqrt{(1-3\,x+4\,x^3)\,(1+3\,x-4\,x^3)}} \\ &= -\frac{3}{\sqrt{1-x^2}} + \frac{3\,(1-4\,x^2)}{\sqrt{(1+x)\,(1-2\,x)^2\,(1-x)\,(1+2\,x)^2}} \\ &= -\frac{3}{\sqrt{1-x^2}} + \frac{3\,(1-4\,x^2)}{\sqrt{(1-x^2)}\,\sqrt{(1-4\,x^2)^2}} \\ &= -\frac{3}{\sqrt{1-x^2}} + \frac{3}{\sqrt{1-x^2}} = 0. \end{aligned} \tag{14}$$

  Note that the cancellation of $(1 - 4\,x^2)$ is only correct when $|x| < 1/2 \Longrightarrow (1 - 4\,x^2) > 0$.

- $h(0) = \pi$.
  We have $\arccos 0 = \frac{\pi}{2}$ therefore $h(0) = \frac{3\,\pi}{2} - \frac{\pi}{2} = \pi$.

Remark. Clearly the above result can be extended to $x = 1/2, -1/2$ through direct calculation.