Math 314 Fall 2013 Homework 7 Solutions

DUE WEDNESDAY NOV. 6 5PM IN ASSIGNMENT BOX (CAB 3RD FLOOR)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $g(x)$ be continuous at $x_0 = 0$. Prove that* $f(x) = \begin{cases} g(x)\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ *is continuous at $x_0 = 0$ if and only if $g(0) = 0$.*

Solution.

1. "If". $g(0) = 0$ then $f(x)$ is continuous at $x_0 = 0$.

   Take any $\varepsilon > 0$. Since $g(x)$ is continuous at $x_0 = 0$, there is $\delta > 0$ such that for all $|x - 0| < \delta$, $|g(x) - 0| < \varepsilon$.

   For these same $|x - 0| < \delta$, we have

   $$|f(x) - f(0)| = \left| g(x)\sin\frac{1}{x} \right| \leqslant |g(x)| < \varepsilon. \tag{1}$$

   Therefore $f(x)$ is continuous at $x_0 = 0$.

2. "Only if". $g(0) \neq 0$ then $f(x)$ is not continuous at $x_0 = 0$.

   We show that in this case $\lim_{x \longrightarrow 0} f(x)$ does not exist, thus $f(x)$ cannot be continuous at 0. Take $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$ for $n \in \mathbb{N}$, we have $x_n, y_n \neq 0$, $x_n \rightarrow 0$, $y_n \rightarrow 0$,

   $$f(x_n) = 0 \longrightarrow 0, \qquad f(y_n) = g(y_n) \longrightarrow g(0) \neq 0. \tag{2}$$

   Thus we have found two subsequences with different limits, and therefore $\lim_{x \longrightarrow 0} f(x)$ does not exist.

Remark. (Other proofs for the "only if" part)

- Method 1. Assume the contrary, that is $\lim_{x \rightarrow 0} f(x) = 0$. Then since $\lim_{x \rightarrow 0} g(x) = g(0) \neq 0$, we have

  $$\lim_{x \rightarrow 0} \sin\frac{1}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{0}{g(0)} = 0 \tag{3}$$

  which contradicts the fact that $\lim_{x \rightarrow 0} \sin\frac{1}{x}$ does not exist.

- Method 2. Assume $f(x)$ is continuous at 0, then for any $x_n \longrightarrow 0$, $x_n \neq 0$, $f(x_n) \longrightarrow f(0) = 0$. Take $x_n = \frac{1}{2n\pi + \pi/2}$. Then $f(x_n) = g(x_n)$. We conclude $g(x_n) \longrightarrow 0$. But g is continuous at $x = 0$, so $g(0) = \lim_{x \longrightarrow 0} g(x) = 0$.

Question 2. *Prove by* ***definition of limit*** *that* $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ *exists and is finite if and only if* $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ *exists and is finite.*

Solution.

- "If".

  Assuming $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists and is finite, we prove $\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite.

  Denote $L := \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

  For any $\varepsilon > 0$, there is $\delta_1 > 0$ such that for any $0 < |h| < \delta_1$,

  $$\left| \frac{f(x_0 + h) - f(x_0)}{h} - L \right| < \varepsilon. \tag{4}$$

Now take $\delta = \delta_1$. For any $0 < |x - x_0| < \delta = \delta_1$, we have

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| = \left|\frac{f(x_0 + (x - x_0)) - f(x_0)}{x - x_0} - L\right| < \varepsilon \tag{5}$$

and therefore $\lim_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0} = L$ is finite.

- "Only if".
  Assuming $\lim_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite, we prove $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists and is finite.
  Denote $L := \mathrm{im}_{h \to 0} \frac{f(x) - f(x_0)}{x - x_0}$.
  For any $\varepsilon > 0$ there is $\delta_2 > 0$ such that for any $0 < |x - x_0| < \delta_2$,

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| < \varepsilon. \tag{6}$$

  Now set $\delta = \delta_2$. For any h satisfying $0 < |h| < \delta$, we have $0 < |(x_0 + h) - x_0| < \delta = \delta_2$. Therefore

$$\left|\frac{f(x_0 + h) - f(x_0)}{h} - L\right| = \left|\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} - L\right| < \varepsilon. \tag{7}$$

  Therefore $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = L$.

Question 3. *Prove that $f(x) = x^3$ is differentiable at every $x_0 \in \mathbb{R}$* ***by definition****.*

Solution. We have

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^3 - x_0^3}{x - x_0} = x^2 + x_0\, x + x_0^2. \tag{8}$$

This is a polynomial of x since x_0 is constant. Therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} [x^2 + x_0\, x + x_0^2] = x_0^2 + x_0^2 + x_0^2 = 3\, x_0^2. \tag{9}$$

So $f(x)$ is differentiable at x_0.

Question 4. *Given $x' = 1$. Use mathematical induction to prove*

$$\forall n \in \mathbb{N}, \qquad (x^n)' = n\, x^{n-1}. \tag{10}$$

Solution. Let $P(n)$ be the statement: $(x^n)' = n\, x^{n-1}$.

- Base is already given.
- $P(n) \Longrightarrow P(n+1)$. Assume $(x^n)' = n\, x^{n-1}$. By Leibniz rule we have

$$(x^{n+1})' = (x^n \cdot x)' = (x^n)' \cdot x + x^n \cdot x' = n\, x^{n-1} \cdot x + x^n = (n+1)\, x^n. \tag{11}$$

Thus ends the proof.

Question 5. *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$. Prove that the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2\, h} \tag{12}$$

exists and equals $f'(x_0)$.

Solution. As $f(x)$ is differentiable at $x_0 \in \mathbb{R}$, there is $\delta > 0$ such that for all $0 < |x - x_0| < \delta$,

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \varepsilon. \tag{13}$$

Now for all $0<|h|<\delta$, set $y:=x_0+h, z=x_0-h$. Then we have

$$0<|y-x_0|<\delta, \qquad 0<|z-x_0|<\delta. \tag{14}$$

This gives through triangle inequality:

$$\begin{aligned}\left|\frac{f(x_0+h)-f(x_0-h)}{2\,h}-f'(x_0)\right| &= \left|\frac{1}{2}\left(\frac{f(y)-f(x_0)}{y-x_0}+\frac{f(z)-f(x_0)}{z-x_0}\right)-f'(x_0)\right| \\ &= \frac{1}{2}\left|\left(\frac{f(y)-f(x_0)}{y-x_0}-f'(x_0)\right)+\left(\frac{f(z)-f(x_0)}{z-x_0}-f'(x_0)\right)\right| \\ &\leqslant \frac{1}{2}\left[\left|\frac{f(y)-f(x_0)}{y-x_0}-f'(x_0)\right|+\left|\frac{f(z)-f(x_0)}{z-x_0}-f'(x_0)\right|\right] \\ &< \frac{1}{2}(\varepsilon+\varepsilon)=\varepsilon. \end{aligned} \tag{15}$$

Thus ends the proof.

Remark. Since "by definition" is not required, it is also OK to prove through:

Since f is differentiable at x_0,

$$\lim_{h\longrightarrow 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0). \tag{16}$$

This gives

$$\lim_{h\longrightarrow 0}\frac{f(x_0-h)-f(x_0)}{-h}=f'(x_0). \tag{17}$$

Therefore

$$\lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}+\frac{f(x_0-h)-f(x_0)}{-h}\right]=2\,f'(x_0). \tag{18}$$

Simplify the LHS we have

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2\,h}=f'(x_0). \tag{19}$$

Question 6. *Let*

$$f(x)=\frac{\exp(x^3)}{\cos x}. \tag{20}$$

Prove that $f(x)$ is differentiable at 0 and calculate $f'(0)$.

Solution.

Since x^3 and e^x are differentiable at every $x\in\mathbb{R}$, so is the composite function $\exp(x^3)$. Furthermore $\cos x$ is differentiable at every $x\in\mathbb{R}$ and $\cos 0=1\neq 0$. So $f(x)$ is differentiable at $x=0$.

We calculate

$$\begin{aligned} f'(x) &= \frac{[\exp(x^3)]'\cos x-\exp(x^3)\,(\cos x)'}{(\cos x)^2} \\ &= \frac{\exp(x^3)\,(x^3)'\cos x+\exp(x^3)\sin x}{(\cos x)^2} \\ &= \frac{3\,x^2\exp(x^3)\cos x+\exp(x^3)\sin x}{(\cos x)^2}. \end{aligned} \tag{21}$$

Setting $x=0$ we have $f'(0)=0$.