Math 314 Fall 2013 Homework 7 Solutions

Due Wednesday Nov. 6 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $g(x)$ be continuous at $x_0=0$. Prove that $f(x)=\begin{cases} g(x)\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ is continuous at $x_0=0$ if and only if $g(0)=0$.*

Question 2. *Prove by **definition of limit** that $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ exists and is finite if and only if $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists and is finite.*

Question 3. *Prove that $f(x)=x^3$ is differentiable at every $x_0\in\mathbb{R}$ **by definition**.*

Question 4. *Given $x'=1$. Use mathematical induction to prove*

$$\forall n\in\mathbb{N},\qquad (x^n)'=n\,x^{n-1}. \tag{1}$$

Question 5. *Let $f(x)$ be differentiable at $x_0\in\mathbb{R}$. Prove that the limit*

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2\,h} \tag{2}$$

exists and equals $f'(x_0)$.

Question 6. *Let*

$$f(x)=\frac{\exp(x^3)}{\cos x}. \tag{3}$$

Prove that $f(x)$ is differentiable at 0 and calculate $f'(0)$.