Math 314 Fall 2013 Homework 6 Solutions

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $f(x) = |x|$. Prove* ***by definition*** *that $f(x)$ is a continuous function (that is $f(x)$ is continuous at every $x_0 \in \mathbb{R}$).*

Solution. For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then for every x satisfying $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = ||x| - |x_0|| \leqslant |x - x_0| < \delta = \varepsilon \tag{1}$$

where the inequality $||x| - |x_0|| \leqslant |x - x_0|$ is triangle inequality. Therefore $f(x)$ is continuous at x_0.

Question 2. *Let $f(x) = \begin{cases} \exp\left[-\frac{1}{x^4}\right] & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove (by definition when necessary) that f is a continuous function.*

Solution. Let $x_0 \in \mathbb{R}$. We prove that $f(x)$ is continuous at x_0. Two cases.

- $x_0 \neq 0$. In this case,
  1. 1 is continuous at x_0; x^4 is continuous at x_0. Furthermore if $x_0 \neq 0$ we have $x_0^4 \neq 0$. Consequently the ratio $\frac{1}{x^4}$ is continuous at x_0.
  2. e^{-x} is a continuous function therefore the composite function $\exp\left[-\frac{1}{|x|}\right]$ is continuous at every $x_0 \neq 0$.
- $x_0 = 0$. We prove by definition. For any $\varepsilon > 0$, there are two cases.
  1. If $\varepsilon < 1$, take $\delta < (-\ln \varepsilon)^{-1/4}$, then we have, for all x satisfying $|x - 0| < \delta$,

  $$|f(x) - f(0)| = \exp\left[-\frac{1}{x^4}\right] < \exp\left(-\frac{1}{\delta}\right) < \varepsilon. \tag{2}$$

  2. **(not required for this homework or midterm, but will be required after midterm)** If $\varepsilon \geqslant 1$, take $\delta = 1$. Then for all x satisfying $|x - 0| < \delta$,

  $$|f(x) - f(0)| = \exp\left[-\frac{1}{x^4}\right] < \exp(-1) < 1 \leqslant \varepsilon. \tag{3}$$

  Therefore $f(x)$ is also continuous at $x_0 = 0$.

Question 3. *Assume there is $\delta_0 > 0$ such that $h(x) \leqslant f(x) \leqslant g(x)$ for all $x \in (x_0 - \delta_0, x_0 + \delta_0)$. Further assume that h, g are continuous at x_0 with $h(x_0) = g(x_0)$. Prove that $f(x)$ is also continuous at x_0.*

Solution. We prove that if there is $\delta_0 > 0$ such that $h(x) \leqslant f(x) \leqslant g(x)$ for all $x \in (x_0 - \delta_0, x_0 + \delta_0)$. Further assume that h, g are continuous at x_0 and $h(x_0) = g(x_0)$, then f is continuous at x_0.

Since h, g are continuous, we have

$$\lim_{x \longrightarrow x_0} h(x) = h(x_0) = g(x_0) = \lim_{x \longrightarrow x_0} g(x). \tag{4}$$

Application of Squeeze theorem gives

$$\lim_{x \longrightarrow x_0} f(x) = h(x_0) = g(x_0). \tag{5}$$

But since $x_0 \in (x_0 - \delta_0, x_0 + \delta_0)$, $h(x_0) \leqslant f(x_0) \leqslant g(x_0)$ together with $h(x_0) = g(x_0) \Longrightarrow f(x_0) = h(x_0) = g(x_0)$. Summarizing, we have

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0) \tag{6}$$

that is $f(x)$ continuous at x_0.

Question 4. *Let* $f(x) = x^6 + 5x^5 - 4x^3 + 10x^2 + 7x - 1$. *Prove that there is* $a \in \mathbb{R}$ *such that* $f(a) = 0$.

Solution. Clearly $f(0) = -1 < 0$. On the other hand we have $f(1) = 18 > 0$. Since $f(x)$ is continuous, it is continuous on $[0, 1]$, by Intermediate Value Theorem we have the existence of $a \in (0, 1)$ such that $f(a) = 0$.

Question 5. *Let* $A, B \subseteq \mathbb{R}$. *Further assume that there is* $m > 0$ *such that for every* $b \in B$, $|b| < m$. *Let* $C := \{a + b|\ a \in A, b \in B\}$. *Prove that* $\sup A - m \leqslant \sup C \leqslant \sup A + m$.

Solution.

- We prove that $\sup A + m$ is an upper bound of C. For any $c \in C$, by definition there are $a \in A, b \in B$ such that $c = a + b < \sup A + m$. Therefore $\sup A + m$ is an upper bound of C and by definition $\sup C \leqslant \sup A + m$.
- We prove $\sup A \leqslant \sup C + m$. For any $a \in A$, take an arbitrary $b \in B$. Then we have

$$a + b \in C \Longrightarrow a + b \leqslant \sup C \Longrightarrow a \leqslant \sup C - b \Longrightarrow a \leqslant \sup C + m. \tag{7}$$

  Thus $\sup C + m$ is an upper bound of A and consequently $\sup A \leqslant \sup C + m \Longrightarrow \sup A - m \leqslant \sup C$.

Question 6. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences of real numbers. Assume* $\lim_{n\to\infty} y_n = 0$. *Prove:*

$$\limsup_{n\longrightarrow\infty} (x_n + y_n) = \limsup_{n\longrightarrow\infty} x_n. \tag{8}$$

Solution. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|y_n| < \varepsilon$. Now for every $k \geqslant n$, we have

$$x_k + y_k \leqslant \sup_{k\geqslant n} x_k + |y_k| \leqslant \sup_{k\geqslant n} x_k + \varepsilon; \tag{9}$$

On the other hand,

$$x_k = x_k + y_k - y_k \leqslant \sup_{k\geqslant n} (x_k + y_k) + |y_k| \leqslant \sup_{k\geqslant n} (x_k + y_k) + \varepsilon. \tag{10}$$

Thus for all $n > N$ we have

$$\sup_{k\geqslant n} x_k - \varepsilon \leqslant \sup_{k\geqslant n} (x_k + y_k) \leqslant \sup_{k\geqslant n} x_k + \varepsilon. \tag{11}$$

Taking limit $n \longrightarrow \infty$, we have

$$\limsup_{n\longrightarrow\infty} x_n - \varepsilon \leqslant \limsup_{n\longrightarrow\infty} (x_n + y_n) < \limsup_{n\longrightarrow\infty} x_n + \varepsilon \tag{12}$$

following Comparison Theorem. Note that this holds for every $\varepsilon > 0$.

Now assume $\limsup_{n\longrightarrow\infty} (x_n + y_n) \neq \limsup_{n\longrightarrow\infty} x_n$. There are two cases.

- $\limsup_{n\longrightarrow\infty} (x_n + y_n) > \limsup_{n\longrightarrow\infty} x_n$. Taking $\varepsilon = \frac{\limsup_{n\longrightarrow\infty} (x_n + y_n) - \limsup_{n\longrightarrow\infty} x_n}{2}$ we have

$$\limsup_{n\longrightarrow\infty} (x_n + y_n) < \limsup_{n\longrightarrow\infty} x_n + \varepsilon = \frac{\limsup_{n\longrightarrow\infty} (x_n + y_n) + \limsup_{n\longrightarrow\infty} x_n}{2} < \limsup_{n\longrightarrow\infty} (x_n + y_n) \tag{13}$$

  contradiction.
- $\limsup_{n\longrightarrow\infty} (x_n + y_n) < \limsup_{n\longrightarrow\infty} x_n$. Taking $\varepsilon = \frac{\limsup_{n\longrightarrow\infty} x_n - \limsup_{n\longrightarrow\infty} (x_n + y_n)}{2}$ we have

$$\limsup_{n\longrightarrow\infty} (x_n + y_n) > \limsup_{n\longrightarrow\infty} x_n - \varepsilon = \frac{\limsup_{n\longrightarrow\infty} (x_n + y_n) + \limsup_{n\longrightarrow\infty} x_n}{2} > \limsup_{n\longrightarrow\infty} (x_n + y_n) \tag{14}$$

  contradiction.

Therefore $\limsup_{n\longrightarrow\infty} (x_n + y_n) = \limsup_{n\longrightarrow\infty} x_n$.

Remark. Alternatively, one can prove as follows. On one hand we have

$$\limsup_{n\to\infty} (x_n + y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n = \limsup_{n\to\infty} x_n + \lim_{n\to\infty} y_n = \limsup_{n\to\infty} x_n; \tag{15}$$

Note that the first inequality has been proved in HW4, and the first equality is because $\lim_{n\to\infty} y_n$ exists;

On the other hand, we have

$$\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} [(x_n + y_n) + (-y_n)] \leqslant \limsup_{n\to\infty} (x_n + y_n) + \limsup_{n\to\infty} (-y_n) = \limsup_{n\to\infty} (x_n + y_n) + \lim_{n\to\infty} (-y_n) = \limsup_{n\to\infty} (x_n + y_n). \tag{16}$$