Math 314 Fall 2013 Homework 5 Solutions

Due Wednesday Oct. 16 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let* $\{x_n\} = \{x_1, x_2, \ldots\}$ *be a **sequence**. Denote*

$$M := \limsup_{n \longrightarrow \infty} x_n, \qquad m := \liminf_{n \longrightarrow \infty} x_n. \tag{1}$$

Critique the following claim:

$$\forall n \in \mathbb{N}, \qquad m - 100 < x_n < M + 100. \tag{2}$$

If it is true provide a proof, otherwise give a counter-example.

Solution. The claim is false. For example, take $x_1 = 200$, $x_2 = -200$ and then $\forall n \in \mathbb{N}$, $n \geqslant 3$, $x_n = 0$. Now clearly $\lim_{n \longrightarrow \infty} x_n = 0$, consequently $M = m = 0$. But $x_1 > M + 100$ and $x_2 < m - 100$.

Question 2. *Are the following series convergent or divergent? Justify your answers.*

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n!}}, \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}. \tag{3}$$

Solution.

- For the first series, apply the ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{\sqrt{n+1}} \tag{4}$$

  thus we have $\lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ which gives $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$. By ratio test we know the series converges.

- For the second series we notice

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{(n+1) - n} = \sqrt{n+1} - \sqrt{n}. \tag{5}$$

  Thus we have

$$s_n := \sum_{k=1}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}} = \sum_{k=1}^{n} \left(\sqrt{k+1} - \sqrt{k} \right) = \sqrt{n+1} - 1. \tag{6}$$

  Since $\lim_{n \longrightarrow \infty} s_n = \infty$, we have by definition of series convergence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \infty. \tag{7}$$

- Althernative method for the 2nd series. We prove:

$$\forall n \geqslant 1, \qquad \frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{3\sqrt{n}}. \tag{8}$$

  All we need to show is $\sqrt{n+1} < 2\sqrt{n} = \sqrt{4n}$ which immediately follows from $4n - (n+1) = 3n - 1 > 0$ for all $n \geqslant 1$. Now since the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty, \tag{9}$$

so does $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$. Consequently $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

Remark. For the second series, both "the series converges to ∞" and "the series diverge to ∞" are correct. If the "alternative" method is used, "the series diverges" is also a correct answer.

Question 3. *Let $x \in \mathbb{R}$. Consider the infinite series*

$$\sum_{n=0}^{\infty} \frac{x^n}{n^{\sqrt{2}}}. \tag{10}$$

Prove that it is convergent when $|x| \leqslant 1$ and divergent when $|x| > 1$.

Solution. We apply the ratio test to $a_n := \frac{x^n}{n^{\sqrt{2}}}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = |x| \left(\frac{n+1}{n}\right)^{\sqrt{2}}. \tag{11}$$

We have

$$\lim_{n \longrightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|. \tag{12}$$

Therefore

$$\operatorname{limsup}_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \operatorname{liminf}_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|. \tag{13}$$

By the ratio test we know the series converges when $|x| < 1$ and diverges when $|x| > 1$.

Now we discuss the case $|x| = 1$. In this case we have

$$|a_n| \leqslant \frac{1}{n^{\sqrt{2}}}. \tag{14}$$

Since $\sum_{n=0}^{\infty} \frac{1}{n^{\sqrt{2}}}$ is convergent, $\sum_{n=0}^{\infty} a_n$ is also convergent when $|x| = 1$.

Question 4. *Calculate the following limits. Provide justification whenever needed.*

$$\lim_{x \longrightarrow 1} \frac{x^2 - 3\,x + 2}{x^2 - 4\,x + 3}, \qquad \lim_{x \longrightarrow \infty} \left(\sqrt[3]{x+5} - \sqrt[3]{x}\right). \tag{15}$$

Solution. We have

$$\lim_{x \longrightarrow 1} \frac{x^2 - 3\,x + 2}{x^2 - 4\,x + 3} = \lim_{x \longrightarrow 1} \frac{(x-2)\,(x-1)}{(x-3)\,(x-1)} = \lim_{x \longrightarrow 1} \frac{x-2}{x-3} = \frac{\lim_{x \longrightarrow 1}(x-2)}{\lim_{x \longrightarrow 1}(x-3)} = \frac{-1}{-2} = \frac{1}{2}. \tag{16}$$

Here the 3rd equality follows from the limit theorem for ratios since both $\lim_{x \to 1}(x-2)$ and $\lim_{x \to 1}(x-3)$ exists and furthermore $\lim_{x \to 1}(x-3) = -2 \neq 0$.

For the second limit we apply the identity $a^3 - b^3 = (a-b)\,(a^2 + a\,b + b^2)$.

$$\lim_{x \to \infty} \left(\sqrt[3]{x+5} - \sqrt[3]{x}\right) = \lim_{x \to \infty} \frac{(x+5) - x}{\sqrt[3]{(x+5)^2} + \sqrt[3]{x+5}\,\sqrt[3]{x} + \sqrt[3]{x^2}} = \lim_{x \to \infty} \frac{5}{\sqrt[3]{(x+5)^2} + \sqrt[3]{x+5}\,\sqrt[3]{x} + \sqrt[3]{x^2}}. \tag{17}$$

Now notice

$$0 \leqslant \frac{5}{\sqrt[3]{(x+5)^2} + \sqrt[3]{x+5}\,\sqrt[3]{x} + \sqrt[3]{x^2}} \leqslant 5\,x^{-2/3} \tag{18}$$

by Squeeze Theorem we have

$$\lim_{x \to \infty} \frac{5}{\sqrt[3]{(x+5)^2} + \sqrt[3]{x+5}\,\sqrt[3]{x} + \sqrt[3]{x^2}} = 0 \tag{19}$$

that is

$$\lim_{x \longrightarrow \infty} \left(\sqrt[3]{x+5} - \sqrt[3]{x}\right) = 0. \tag{20}$$

Question 5. *Discuss the existence/non-existence of the following limits. If a limit exists find the limit and justify your calculation, otherwise provide a proof.*

$$\lim_{x \longrightarrow \infty} \exp[\sin x + 1], \qquad \lim_{x \longrightarrow \infty} \exp[\sin x - 3x] \tag{21}$$

Solution. The first limit does not exist. Take $x_n = 2n\pi$ and $y_n = \left(2n + \frac{1}{2}\right)\pi$. We have

$$\sin x_n = 0, \qquad \sin y_n = 1. \tag{22}$$

Now check

$$\lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} y_n = \infty; \tag{23}$$

$$\lim_{x \longrightarrow \infty} \exp[\sin x_n + 1] = e, \qquad \lim_{x \longrightarrow \infty} \exp[\sin y_n + 1] = e^2 \tag{24}$$

They are different so the limit does not exist.

The second limit is 0. Since $\sin x \leqslant 1$, we have

$$0 \leqslant \exp[\sin x - 3x] \leqslant \exp[1 - 3x] = e\, e^{-3x}. \tag{25}$$

Applying Squeeze Theorem we conclude

$$\lim_{x \longrightarrow \infty} \exp[\sin x - 3x] = 0. \tag{26}$$

Question 6. *Let $f, g: \mathbb{R} \mapsto \mathbb{R}$ be functions. Let $a \in \mathbb{R}$. Critique the following claim:*

If $\lim_{x \longrightarrow a} f(x) = b$ and $\lim_{x \longrightarrow b} g(x) = L$, then $\lim_{x \longrightarrow a} g(f(x)) = L$.

If it is true provide a proof, otherwise find a counter-example.

Solution. The claim is false. For example, let $a = 0$, $f(x) = 1$ and $g(x) = \left\{ \begin{array}{ll} 0 & x = 1 \\ 1 & x \neq 0 \end{array} \right.$. Then $\lim_{x \longrightarrow 0} f(x) = 1$, $\lim_{x \longrightarrow 1} g(x) = 1$ but $g(f(x)) = 0$ so $\lim_{x \longrightarrow 0} g(f(x)) = 0 \neq 1$.