Math 314 Fall 2013 Homework 4 Solutions

DUE WEDNESDAY OCT. 9 5PM IN ASSIGNMENT BOX (CAB 3RD FLOOR)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.
- You can use any theorem/lemma/proposition in the lecture notes (Please explici

Question 1. *Let $f: X \mapsto Y$ be a function. Critique the following claim.*

f is one-to-one if and only if $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of X.

If it is true prove it; Otherwise provide a counter-example.

Solution. True.

- "If". Assume that f is not one-to-one. Then there are $x_1, x_2 \in X$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Take $A = \{x_1\}$, $B = \{x_2\}$, then $A \cap B = \varnothing$ so $f(A \cap B) = \varnothing$. But $f(A) \cap f(B) = \{f(x_1)\} \neq \varnothing$. Contradiction.
- "Only if". Assume that there are $A, B \subseteq X$ such that $f(A \cap B) \neq f(A) \cap f(B)$. Since $f(A \cap B) \subseteq f(A) \cap f(B)$, there is $y \in Y$ such that $y \in f(A) \cap f(B)$ but $y \notin f(A \cap B)$. As $y \in f(A)$, there is $x_1 \in A$ such that $y = f(x_1)$; As $y \in f(B)$ there is $x_2 \in B$ such that $y = f(x_2)$. Because $y \notin f(A \cap B)$, $x_1 \neq x_2$. This contradicts f being one-to-one.

Question 2. *Let $x_0 \in \mathbb{R}$ be an arbitrary real number different from 2 and define x_n through*

$$x_n = \frac{x_{n-1}}{2} + 1. \tag{1}$$

Does the sequence converge? If so find the limit. Justify your answer.

Solution. We have

$$|x_{n+1} - x_n| = \frac{|x_n - x_{n-1}|}{2} \Longrightarrow |x_{n+1} - x_n| = \left(\frac{1}{2}\right)^n |x_1 - x_0|. \tag{2}$$

For any $\varepsilon > 0$, since $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ converges, there is $N \in \mathbb{N}$ such that for all $n > m > N$,

$$\left| \sum_{k=m}^{n-1} \left(\frac{1}{2}\right)^k \right| < \frac{\varepsilon}{|x_1 - x_0|}. \tag{3}$$

Note that since $x_0 \neq 2$, $x_1 - x_0 \neq 0$.

This means for any $n > m > N$,

$$|x_n - x_m| \leqslant |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| = \left[\left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^m\right] |x_1 - x_0| < \varepsilon. \tag{4}$$

Thus $\{x_n\}$ is Cauchy and there is $x \in \mathbb{R}$ such that $x_n \longrightarrow x$. Therefore

$$x = \lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} \left(\frac{x_{n-1}}{2} + 1\right) = \frac{x}{2} + 1 \Longrightarrow x = 2. \tag{5}$$

Remark. Alternatively, one can solve x_n one by one:

$$x_n = \frac{x_{n-1}}{2} + 1 = \frac{\frac{x_{n-2}}{2} + 1}{2} + 1 = \frac{x_{n-2}}{2^2} + 2^{-1} + 1 = \frac{x_{n-3}}{2^3} + 2^{-2} + 2^{-1} + 1 = \cdots = \frac{x_0}{2^n} + 2^{-(n-1)} + \cdots + 1 \tag{6}$$

and then take limit.

Remark. Alternatively, one can prove that:

- If $x_0 < 2$, then x_n is increasing and upper bounded by 2;

- If $x_0 > 2$, then x_n is decreasing and lower bounded by 2.

and then conclude that the limit exists.

Question 3. *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be non-negative series with* $a_n > 0, b_n > 0$ *for all* $n \in \mathbb{N}$.

a) **(3 pts)** *If* $\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$, *then* $\sum_{n=1}^{\infty} b_n$ *converges* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *converges;*

b) **(2 pt)** *Use a) to prove convergence for* $\sum_{n=1}^{\infty} a_n$ *with* $a_1 = 1$ *and*

$$a_n = \frac{1}{4} \frac{2}{5} \cdots \frac{n-1}{n+2} \tag{7}$$

(Hint: use $b_n = \frac{1}{n(n+1)}$. *)*

Solution.

a) From the assumption we have (note that $a_1 > 0$ is used here)

$$\frac{a_2}{a_1} \leqslant \frac{b_2}{b_1} \Longrightarrow a_2 \leqslant \frac{a_1}{b_1} b_2; \tag{8}$$

$$\frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \leqslant \frac{b_3}{b_2} \cdot \frac{b_2}{b_1} \Longrightarrow \frac{a_3}{a_1} \leqslant \frac{b_3}{b_1} \Longrightarrow a_3 \leqslant \frac{a_1}{b_1} b_3; \tag{9}$$

In general we have

$$a_n \leqslant \frac{a_1}{b_1} b_n \tag{10}$$

for all $n \in \mathbb{N}$.

Now for any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} b_n$ converges, there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} b_k \right| < \frac{b_{N_0}}{a_{N_0}} \varepsilon. \tag{11}$$

Take $N = \max\{N_0, N_1\}$. We have for all $m > n > N$, (Note that we need the positivity of a_k in the first inequality below)

$$\left| \sum_{k=n+1}^{m} a_k \right| \leqslant \left| \sum_{k=n+1}^{m} \frac{a_{N_0}}{b_{N_0}} b_k \right| = \frac{a_{N_0}}{b_{N_0}} \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon. \tag{12}$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges.

b) Take b_n as in the hint. We know that $\sum_{n=1}^{\infty} b_n = 1$. Now check

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+3} \leqslant \frac{b_{n+1}}{b_n} = \frac{n}{n+2} \tag{13}$$

for all $n \geqslant 2$. Thus application of a) gives the convergence of $\sum_{n=1}^{\infty} a_n$.

Remark. In fact this problem is a bit silly as

$$a_n = \frac{(n-1)!}{[(n+2)!/6]} = \frac{6}{(n+2)(n+1)n}. \tag{14}$$

Question 4. *Let* $\{x_n\}$, $\{y_n\}$ *be sequences of real numbers. Which of the following is the most precise relation between* $\limsup_{n \longrightarrow \infty} (x_n + y_n)$ *and* $\limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$?

a) $\limsup_{n \to \infty} (x_n + y_n) = \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

b) $\limsup_{n \to \infty} (x_n + y_n) \leqslant \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

c) $\limsup_{n \to \infty} (x_n + y_n) \leqslant \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$ *and it may happen that* $\limsup_{n \to \infty} (x_n + y_n) < \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

Justify your answer.

Solution. The most precise relation is c).

- We prove $\limsup_{n\to\infty}(x_n+y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$. By a theorem in lecture notes, we have (recall that a sequence is in fact a function with domain $\mathbb{N}$)
$$\sup_{k\geqslant n}(x_k+y_k) \leqslant \sup_{k\geqslant n} x_k + \sup_{k\geqslant n} y_k \tag{15}$$
Taking limit, by Comparison Theorem, we have $\limsup_{n\to\infty}(x_n+y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.
- An example of $\limsup_{n\to\infty}(x_n+y_n) < \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$. Take $x_n = (-1)^n$, $y_n = (-1)^{n+1}$. Then $\limsup x_n = \limsup y_n = 1$ but $x_n + y_n = 0$ for all $n \in \mathbb{N}$ so $\limsup_{n\to\infty}(x_n+y_n) = 0 < 1+1$.

Question 5. *Let $\{x_n\}$ be a sequence and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Prove that if $\{x_{n_k}\}$ is not bounded above, then $\{x_n\}$ is not bounded above either.*

Proof. First we obtain the negation of $\{x_n\}$ is bounded above:
$$\neg[\exists M \in \mathbb{R}\ \forall n \in \mathbb{N} \quad x_n \leqslant M] = [\forall M \in \mathbb{R}\ \exists n \in \mathbb{N} \quad x_n > M]. \tag{16}$$
Thus the assumption $\{x_{n_k}\}$ is not bounded above means
$$\forall M \in \mathbb{R}\ \exists k \in \mathbb{N} \quad x_{n_k} > M. \tag{17}$$
Thus for any $M \in \mathbb{R}$, there is $k \in \mathbb{N}$ such that $x_{n_k} > M$. Now take $n = n_k$. We see that there is $n \in \mathbb{N}$ such th at $x_n > M$. Therefore $\forall M \in \mathbb{R}\ \exists n \in \mathbb{N} \quad x_n > M$ and $\{x_n\}$ is not bounded above. □

Remark. Alternatively, one can prove by contradiction. Assume $\{x_n\}$ is bounded above. Then there is $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $x_n \leqslant M$. Now for every $k \in \mathbb{N}$, $n_k \in \mathbb{N}$ and therefore $x_{n_k} \leqslant M$. So $\{x_{n_k}\}$ is bounded above. Contradiction.

Question 6. *Prove* $\sum_{n=1}^{\infty} \frac{1}{n \log_2(n+1)} = \infty$.

Proof. We have
$$\begin{aligned}
\frac{1}{1\log_2(1+1)} &> \frac{1}{2}; && (18)\\
\frac{1}{2\log_2(2+1)} + \frac{1}{3\log_2(3+1)} &> \frac{1}{4\log_2 4} + \frac{1}{4\log_2 4} = \frac{1}{4}; && (19)\\
\frac{1}{4\log_2(4+1)} + \cdots + \frac{1}{7\log_2(7+1)} &> \frac{4}{8\log_2 8} = \frac{1}{6}; && (20)\\
\vdots \quad &\vdots \quad \vdots\\
\frac{1}{2^{n-1}\log_2(2^{n-1}+1)} + \frac{1}{(2^n-1)\log_2(2^n)} &> \frac{2^{n-1}}{2^n n} = \frac{1}{2n}; && (21)\\
\vdots \quad &\vdots \quad \vdots
\end{aligned}$$

Therefore
$$S_{2^n-1} := \sum_{k=1}^{2^n-1} \frac{1}{k\log_2(k+1)} > \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}. \tag{22}$$
Now for any $M > 0$, since $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}$ is not bounded from above, there is $n_0 \in \mathbb{N}$ such that $\sum_{k=1}^{n_0} \frac{1}{k} > M$. This gives
$$S_{2^{n_0}-1} > M \tag{23}$$
and therefore $\{S_n\}$ is not bounded from above which means $\sum_{n=1}^{\infty} \frac{1}{n\log_2(n+1)} = \infty$. □