Math 314 Fall 2013 Homework 4

Due Wednesday Oct. 9 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $f: X \mapsto Y$ be a function. Critique the following claim.*

f is one-to-one if and only if $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of X.

If it is true prove it; Otherwise provide a counter-example.

Question 2. *Let $x_0 \in \mathbb{R}$ be an arbitrary real number different from 2 and define x_n through*

$$x_n = \frac{x_{n-1}}{2} + 1. \tag{1}$$

Does the sequence converge? If so find the limit. Justify your answer.

Question 3. *Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be non-negative series with $a_n > 0, b_n > 0$ for all $n \in \mathbb{N}$.*

a) **(3 pts)** *If $\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$, then $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges;*

b) **(2 pt)** *Use a) to prove convergence for $\sum_{n=1}^{\infty} a_n$ with $a_1 = 1$ and*

$$a_n = \frac{1}{4} \frac{2}{5} \cdots \frac{n-1}{n+2} \tag{2}$$

(Hint: use $b_n = \frac{1}{n\,(n+1)}$.)

Question 4. *Let $\{x_n\}$, $\{y_n\}$ be sequences of real numbers. Which of the following is the most precise relation between $\limsup_{n \longrightarrow \infty} (x_n + y_n)$ and $\limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$?*

a) $\limsup_{n \to \infty} (x_n + y_n) = \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

b) $\limsup_{n \to \infty} (x_n + y_n) \leqslant \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

c) $\limsup_{n \to \infty} (x_n + y_n) \leqslant \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$ *and it may happen that* $\limsup_{n \to \infty} (x_n + y_n) < \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n$.

Justify your answer.

Question 5. *Let $\{x_n\}$ be a sequence and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Prove that if $\{x_{n_k}\}$ is not bounded above, then $\{x_n\}$ is not bounded above either.*

Question 6. *Prove $\sum_{n=1}^{\infty} \frac{1}{n \log_2 (n+1)} = \infty$.*