Math 314 Fall 2013 Homework 3

Due Wednesday Oct. 2 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let* $x_n = (-1)^n - e^{-n}$ *and* $E = \{x_n : n \in \mathbb{N}\}$. *(*$\mathbb{N} = \{1, 2, 3, ...\}$*). Find* $\max E, \sup E, \min E, \inf E$. *Justify your answers.*

Question 2. *Let* $f : X \mapsto Y$ *be a function. Let* $A, B \subseteq X$ *and* $S, T \subseteq Y$.

a) Prove: If $A \subseteq B$ *then* $f(A) \subseteq f(B)$.

b) Prove: If $S \subseteq T$ *then* $f^{-1}(S) \subseteq f^{-1}(T)$.

c) Is it true that $A \subset B$ *implies* $f(A) \subset f(B)$*? Justify your answer.*

d) Is it true that $S \subset T$ *implies* $f^{-1}(S) \subset f^{-1}(T)$*? Justify your answer.*

Question 3. *Let* $A \subseteq X, B \subseteq Y$ *and* $f : X \mapsto Y$. *Prove that*

a) $f(f^{-1}(B)) \subseteq B$.

b) $f^{-1}(f(A)) \supseteq A$.

c) If $B \subseteq f(X)$*, then* $f(f^{-1}(B)) = B$.

Question 4. *Let* $x_n = n^a$ *for* $a \in \mathbb{R}$. *Discuss whether* $\lim_{n \longrightarrow \infty} n^a$ *exists or not. If it exists find and prove the limit. If it does not prove that the limit does not exist.*

Question 5. *Calculate the limits of the following sequences.*

$$x_n = \frac{100\, n^2 - 2\, n^4}{n^4 + 3\, n}, \quad y_n = \sqrt{n+1} - \sqrt{n-1}, \quad z_n = \frac{\sin n^3}{n}. \tag{1}$$

Justify your answers. You can use the results from the previous question.

Question 6. *Let* $0 < y_1 < x_1$ *and set*

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n\, y_n}, \qquad n \in \mathbb{N}. \tag{2}$$

a) Prove that $0 < y_n < x_n$ *for all* $n \in \mathbb{N}$*;*

b) Prove that y_n *is increasing and bounded above, and* x_n *is decreasing and bounded below;*

c) Prove that $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/(2^n)$ *for all* $n \in \mathbb{N}$*;*

d) Prove that $\lim_{n \longrightarrow \infty} x_n$, $\lim_{n \longrightarrow \infty} y_n$ *both exist and are equal.*