Math 314 Fall 2013 Homework 2 Solutions

DUE WEDNESDAY SEPT. 25 5PM IN ASSIGNMENT BOX (CAB 3RD FLOOR)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let $E \subseteq \mathbb{R}$. Prove that $(E^c)^c = E$.*

Proof. We need to show $(E^c)^c \subseteq E$ and $E \subseteq (E^c)^c$.

- $(E^c)^c \subseteq E$.
  Take any $x \in (E^c)^c$. By definition of complement, we have $x \notin E^c$. Now if $x \notin E$, by definition of E^c, $x \in E^c$. Contradiction. Therefore $x \in E$.
- $E \subseteq (E^c)^c$.
  Take any $x \in E$. If $x \in E^c$ then by definition $x \notin E$, contradiction. Therefore $x \notin E^c$ and by definition of complement, $x \in (E^c)^c$. □

Question 2. *Let $A, B \subseteq \mathbb{R}$. Prove that*

a) $(A \cap B)^c = A^c \cup B^c$;

b) $(A \cup B)^c = A^c \cap B^c$.

Proof.

a) Two steps.

1. $(A \cap B)^c \subseteq A^c \cup B^c$.
   Take any $x \in (A \cap B)^c$. By definition $x \notin A \cap B$. Now there are two cases.
   i. $x \in A$. We claim that $x \notin B$. Assume the contrary. Then $x \in B$. Since $x \in A$ too, $x \in A \cap B$. Contradiction.
   ii. $x \notin A$. Then $x \in A^c \subseteq A^c \cup B^c$.
2. $A^c \cup B^c \subseteq (A \cap B)^c$.
   Take any $x \in A^c \cup B^c$. There are two cases:
   - $x \in A^c$. This gives $x \notin A \Longrightarrow x \notin A \cap B \Longrightarrow x \in (A \cap B)^c$;
   - $x \in B^c$. This gives $x \notin B \Longrightarrow x \notin A \cap B \Longrightarrow x \in (A \cap B)^c$.

b) Two steps.

1. $(A \cup B)^c \subseteq A^c \cap B^c$.
   Take any $x \in (A \cup B)^c$. Then $x \notin A \cup B$. As $A \subseteq A \cup B$, $x \notin A \cup B \Longrightarrow x \notin A \Longrightarrow x \in A^c$; As $B \subseteq A \cup B$, $x \notin A \cup B \Longrightarrow x \notin B \Longrightarrow x \in B^c$. Therefore $x \in A^c \cap B^c$.
2. $A^c \cap B^c \subseteq (A \cup B)^c$.
   Take any $x \in A^c \cap B^c$. We prove by contradiction. Assume $x \notin (A \cup B)^c$. Then $x \in (A \cup B)$ as proved in Question 1. Two cases.
   - $x \in A$. Then $x \notin A^c \Longrightarrow x \notin A^c \cap B^c \subseteq A^c$. Contradiction.
   - $x \in B$. Then $x \notin B^c \Longrightarrow x \notin A^c \cap B^c \subseteq B^c$. Contradiction. □

Question 3. *Find infinitely many nonempty sets of natural numbers*

$$\mathbb{N} \supset S_1 \supset S_2 \supset \cdots \tag{1}$$

such that $\cap_{n=1}^{\infty} S_n = \varnothing$. You need to rigorously justify your claim.

Solution. Take $S_n = \{m \in \mathbb{N} : m > n\} = \{n+1, n+2, \ldots\}$.

- First show S_n is nonempty. By construction we have $n+1 \in S_n$ so it is nonempty.
- Next show $\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$. By definition $S_1 \subseteq \mathbb{N}$. As $1 \in \mathbb{N}$, $1 \notin S_1$ we have $S_1 \subset \mathbb{N}$. Next we show $S_{k+1} \subset S_k$ for every $k \in \mathbb{N}$. Take any $m \in S_{k+1}$. By definition of S_{k+1} we must have $m > k+1 > k$ therefore $m \in S_k$. So $S_{k+1} \subseteq S_k$. Since $k+1 \in S_k$, $k+1 \notin S_{k+1}$, we have $S_k \neq S_{k+1}$. Therefore $S_{k+1} \subset S_k$.
- Finally show $\cap_{n=1}^{\infty} S_n = \varnothing$. We prove by contradiction. Assume $\cap_{n=1}^{\infty} S_n \neq \varnothing$. Then there is $m \in \cap_{n=1}^{\infty} S_n$. However by construction of S_m, $m \notin S_m$. Contradiction.

Question 4. *Prove* ***by definition****:*

a) $(0,1) \cup (2,3)$ *is open;*

b) $[0,1] \cup [7,8]$ *is closed.*

Proof.

a) Let $x \in (0,1) \cup (2,3)$ be arbitrary. Two cases.

- $x \in (0,1)$. The open interval $(0,1)$ satisfies
$$x \in (0,1) \subseteq (0,1) \cup (2,3). \tag{2}$$
- $x \in (2,3)$. The open interval $(2,3)$ satisfies
$$x \in (2,3) \subseteq (0,1) \cup (2,3). \tag{3}$$

b) First we have
$$([0,1] \cup [7,8])^c = (-\infty, 0) \cup (1,7) \cup (8, \infty). \tag{4}$$
We will prove that this set is open. Take an arbitrary $x \in ([0,1] \cup [7,8])^c$. Three cases.

- $x \in (-\infty, 0)$. The open interval $(-\infty, 0)$ satisfies
$$x \in (-\infty, 0) \subseteq (-\infty, 0) \cup (1,7) \cup (8, \infty); \tag{5}$$
- $x \in (1,7)$. The open interval $(1,7)$ satisfies
$$x \in (1,7) \subseteq (-\infty, 0) \cup (1,7) \cup (8, \infty); \tag{6}$$
- $x \in (8, \infty)$. The open interval $(8, \infty)$ satisfies
$$x \in (8, \infty) \subseteq (-\infty, 0) \cup (1,7) \cup (8, \infty). \tag{7}$$

□

Question 5. *Let* $E := \{(-1)^n + e^{-n} : n \in \mathbb{N}\}$. *Find* $\max E, \sup E, \min E, \inf E$. *Justify your answers.*

Solution.

- $\max E = 1 + e^{-2}$. To justify, we show that
  1. $1 + e^{-2} \in E$. This is clear since $1 + e^{-2} = (-1)^2 + e^{-2}$.
  2. $\forall a \in E, 1 + e^{-2} \geqslant a$. Since $a \in E$, there is $n \in \mathbb{N}$ such that $a = (-1)^n + e^{-n}$. There are two cases:
     a) n odd. In this case $a = -1 + e^{-n} \leqslant -1 + 1 = 0 < 1 + e^{-2}$.
     b) n even. In this case $a = 1 + e^{-n} \leqslant 1 + e^{-2}$ since $n \geqslant 2$.
- Since $\max E$ exists, we have $\sup E = \max E = 1 + e^{-2}$.
- $\min E$ does not exist. To see this, assume the contrary. Then there is $n_0 \in \mathbb{N}$ such that $(-1)^{n_0} + e^{-n_0} \leqslant (-1)^n + e^{-n}$ for all $n \in \mathbb{N}$. Take $n = n_0 + 2$. We have
$$(-1)^n + e^{-n} = (-1)^{n_0} + e^{-n_0 - 2} < (-1)^{n_0} + e^{-n_0} \tag{8}$$

Contradiction.

- $\inf E = -1$. To justify, we need to show
  1. $-1 \leqslant (-1)^n + e^{-n}$ for all $n \in \mathbb{N}$. We have $(-1)^n + e^{-n} \geqslant (-1)^n \geqslant -1$ for all $n \in \mathbb{N}$ so this part is proved.
  2. Any $b > -1$ is not a lower bound. Let $b > -1$ be arbitrary. Take $n > -\ln(b+1)$, then
  $$(-1)^{2n+1} + e^{-(2n+1)} < -1 + e^{-n} < -1 + (b+1) = b. \tag{9}$$
  Therefore b is not a lower bound of E.

Question 6. *Let A, $B \subseteq \mathbb{R}$. Define their sum as the set $A + B := \{x + y | \ x \in A, \ y \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$, $\inf(A+B) = \inf A + \inf B$.*

Proof. To show $\sup(A+B) = \sup A + \sup B$, we need to show

1. $\sup A + \sup B \geqslant z$ for every $z \in A + B$. Take any $z \in A + B$. Then there are $x \in A$, $y \in B$ such that $z = x + y$. By definition of sup we have $\sup A \geqslant x, \sup B \geqslant y$. Consequently $\sup A + \sup B \geqslant z$.
2. If $b \in \mathbb{R}$ satisfies $b \geqslant z$ for every $z \in A + B$, then $b \geqslant \sup A + \sup B$. By a theorem in the lecture notes, for every $\varepsilon > 0$, there is $x \in A, y \in B$ such that $x > \sup A - \varepsilon/2, y > \sup B - \varepsilon/2$. Thus for every $\varepsilon > 0$, $b \geqslant x + y > \sup A + \sup B - \varepsilon$. Consequently $b \geqslant \sup A + \sup B$.

To show $\inf(A+B) = \inf A + \inf B$, we need to show

1. $\inf A + \inf B \leqslant z$ for every $z \in A + B$. Take any $z \in A + B$, there are $x \in A, y \in B$ such that $z = x + y$. By definition of inf we have $x \geqslant \inf A, y \geqslant \inf B$. So $\inf A + \inf B \leqslant z$.
2. If $b \in \mathbb{R}$ satisfies $b \leqslant z$ for every $z \in A + b$, then $b \leqslant \inf A + \inf B$. By a theorem in the lecture notes, for every $\varepsilon > 0$, there is $x \in A, y \in B$ such that $x < \inf A + \varepsilon/2, y < \inf B + \varepsilon/2$. Thus for every $\varepsilon > 0$, $b \leqslant x + y < \inf A + \inf B + \varepsilon$. Consequently $b \leqslant \inf A + \inf B$. $\square$