Math 314 Fall 2013 Homework 10 Solutions

Due Wednesday Nov. 27 5pm in Assignment Box (CAB 3rd Floor)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let*

$$f(x)=\begin{cases} 1 & x=1 \\ 0 & x\neq 1 \end{cases}. \tag{1}$$

Prove by definition that $f(x)$ is Riemann integrable on $[0,2]$.

Solution. For any partition $P=\{0=x_0<x_1<\cdots<x_n=2\}$, we have

$$\forall i=0,\ldots,n-1, \qquad \inf_{x\in[x_i,x_{i+1}]} f(x)=0. \tag{2}$$

Therefore $L(f,P)=0$ for all P and consequently $L(f)=0$.

On the other hand, for any $n\in\mathbb{N}$, take the particular partition $P_n=\left\{0,\frac{n-1}{n},\frac{n+1}{n},1\right\}$. Then we have

$$\sup_{x\in\left[0,\frac{n-1}{n}\right]} f(x)=0, \qquad \sup_{x\in\left[\frac{n-1}{n},\frac{n+1}{n}\right]} f(x)=1, \qquad \sup_{x\in\left[\frac{n+1}{n},1\right]} f(x)=0. \tag{3}$$

Therefore

$$U(f,P_n)=\frac{2}{n}. \tag{4}$$

By defintion

$$U(f)\leqslant\frac{2}{n} \tag{5}$$

for all $n\in\mathbb{N}$. Taking limit $n\longrightarrow\infty$, we have

$$U(f)\leqslant 0 \tag{6}$$

thanks to Comparison Theorem.

Recall that $U(f)\geqslant L(f)$. Thus we reach $L(f)=U(f)=0$ and integrability follows.

Question 2. *Let $f(x),g(x)$ be integrable functions on $[a,b]$. Prove by definition that if $f(x)\leqslant g(x)$ for all $x\in[a,b]$, then $\int_a^b f(x)\,\mathrm{d}x\leqslant\int_a^b g(x)\,\mathrm{d}x$.*

Solution. Let P be an arbitrary partition of $[a,b]$. Then we have

$$U(f,P)=\sum_{j=1}^n\left[\sup_{[x_{j-1},x_j]} f(x)\right]|x_j-x_{j-1}|\leqslant\sum_{j=1}^n\left[\sup_{[x_{j-1},x_j]} g(x)\right]|x_j-x_{j-1}|=U(g,P). \tag{7}$$

From this we have

$$U(f)\leqslant U(f,P)\leqslant U(g,P) \tag{8}$$

for any partition P. Thus $U(f)$ is a lower bound for the set

$$\{U(g,P)|\,P\text{ is a partition of }[a,b]\}. \tag{9}$$

By definition $U(f)$ is the infimum of this set and is thus the greatest lower bound for it. Therefore

$$U(f) \leqslant U(g). \tag{10}$$

Since f, g are integrable, this gives $\int_a^b f(x)\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$.

Question 3. *It is true that $|f(x)|$ is integrable on $[a,b] \Longrightarrow f(x)$ integrable on $[a,b]$? Justify your answer.*

Solution. $|f|$ integrable $\Longrightarrow f$ integrable is false. An example is

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases}, \qquad [a,b] = [0,1]. \tag{11}$$

We have $|f(x)| = 1$ constant, so $|f|$ is integrable on $[a,b]$.

On the other hand, over any interval we always have $\sup f = 1, \inf f = -1$ therefore

$$U(f,P) = 1, \qquad L(f,P) = -1 \tag{12}$$

for any partition P. This gives $U(f) = 1 \neq -1 = L(f)$ which means f is not integrable.

Question 4. *Calculate the following integrals through integration by parts or change of variable.*

$$I_1 = \int_0^\pi e^x \sin x\,\mathrm{d}x; \qquad I_2 = \int_1^e x \ln x\,\mathrm{d}x; \qquad I_3 = \int_1^2 \frac{\mathrm{d}x}{e^x + e^{-x}} \tag{13}$$

Solution.

- I_1. We have

$$\begin{aligned} I_1 &= \int_0^\pi \sin x\,(e^x)'\,\mathrm{d}x \\ &= [e^x \sin x]_0^\pi - \int_0^\pi e^x\,(\sin x)'\,\mathrm{d}x \\ &= -\int_0^\pi e^x \cos x\,\mathrm{d}x \\ &= -\int_0^\pi \cos x\,(e^x)'\,\mathrm{d}x \\ &= -[e^x \cos x]_{x=0}^{x=\pi} + \int_0^\pi e^x\,(\cos x)'\,\mathrm{d}x \\ &= 1 + e^\pi - \int_0^\pi e^x \sin x\,\mathrm{d}x = 1 + e^\pi - I_1. \end{aligned} \tag{14}$$

  Therefore $I_1 = \frac{1+e^\pi}{2}$.

- I_2. We have

$$I_2 = \int_1^e (\ln x)\left(\frac{x^2}{2}\right)'\mathrm{d}x = \left[\frac{x^2}{2}\ln x\right]_{x=1}^{x=e} - \int_1^e \frac{x^2}{2}\,(\ln x)'\,\mathrm{d}x = \frac{e^2}{2} - \int_1^e \frac{x}{2}\,\mathrm{d}x = \frac{(e^2+1)}{4}. \tag{15}$$

- I_3. Set $t = u(x) = e^x$. We have

$$I_2 = \int_1^2 \frac{e^x\,\mathrm{d}x}{1+(e^x)^2} = \int_e^{e^2} \frac{\mathrm{d}t}{1+t^2} = \arctan(e^2) - \arctan(e). \tag{16}$$

Question 5. *Let f be continuous on $[a,b]$. Let $G(x) = \int_{-x}^{\sin x} f(t)\,\mathrm{d}t$. Calculate $G'(x)$. Justify your answer. (Hint: define $F(x) = \int_0^x f(t)\,\mathrm{d}t$ and use F to represent $G(x)$.)*

Solution. We have

$$G(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t + \int_{-x}^0 f(t)\,\mathrm{d}t = \int_0^{\sin x} f(t)\,\mathrm{d}t - \int_0^{-x} f(t)\,\mathrm{d}t = F(\sin x) - F(-x). \tag{17}$$

Now by chain rule:

$$G'(x) = F'(\sin x)\cos x - F'(-x)\,(-1). \tag{18}$$

Since $f(x)$ is continuous, by FTC we have

$$F'(x) = f(x). \tag{19}$$

Therefore

$$G'(x) = f(\sin x)\cos x + f(-x). \tag{20}$$

Question 6. *Prove that the improper integral*

$$\int_0^\infty e^{-2x}\cos(3\,x)\,\mathrm{d}x \tag{21}$$

exists and calculate its value.

Solution. Notice that $e^{-2x}\cos(3\,x)$ is continuous on $[0,c]$ for every $c>0$ and is therefore integrable there, we calculate

$$\begin{aligned}\int_0^c e^{-2x}\cos(3\,x)\,\mathrm{d}x &= \int_0^c e^{-2x}\left(\frac{1}{3}\sin(3\,x)\right)'\mathrm{d}x\\ &= e^{-2c}\frac{1}{3}\sin(3\,c) - e^{-2\cdot 0}\frac{1}{3}\sin(3\cdot 0)\\ &\quad -\int_0^c \frac{1}{3}\sin(3\,x)\,(e^{-2x})'\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3\,c) + \frac{2}{3}\int_0^c e^{-2x}\sin(3\,x)\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3\,c) - \frac{2}{9}\int_0^c e^{-2x}(\cos(3\,x))'\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3\,c) - \frac{2}{9}\left[e^{-2c}\cos(3\,c) - e^{-2\cdot 0}\cos(3\cdot 0) + 2\int_0^c e^{-2x}\cos(3\,x)\,\mathrm{d}x\right]\\ &= \frac{1}{3}e^{-2c}\sin(3\,c) - \frac{2}{9}e^{-2c}\cos(3\,c) + \frac{2}{9} - \frac{4}{9}\int_0^c e^{-2x}\cos(3\,x)\,\mathrm{d}x. \end{aligned}\tag{22}$$

Thus

$$\int_0^c e^{-2x}\cos(3\,x)\,\mathrm{d}x = \frac{9}{13}\left[\frac{1}{3}e^{-2c}\sin(3\,c) - \frac{2}{9}e^{-2c}\cos(3\,c) + \frac{2}{9}\right]. \tag{23}$$

Taking limit $c \longrightarrow \infty$ we have

$$\lim_{c\longrightarrow\infty}\int_0^c e^{-2x}\cos(3\,x)\,\mathrm{d}x = \frac{2}{13} \tag{24}$$

exists and is finite. By Theorem 44, the improper integral exists, and equals

$$\int_0^\infty e^{-2x}\cos(3\,x)\,\mathrm{d}x = \frac{2}{13}. \tag{25}$$