Sets and Functions

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1. Sets.

Definition 1. (Set) *A set is a collection of objects. These objects are called "elements" or "members" of the set.*

If an object x is an element of a set A, we write $x \in A$, or equivalently $A \ni x$. If an object is not an element of a set A, we write $x \notin A$.

There are two ways of defining a set.

- The first is to list all elements:

$$A := \{1, 2, 3\}; \qquad B := \{1, \{2, 3\}\} \tag{1}$$

  Note that A and B are different!

  Exercise 1. Explain why A and B are different.

  On the other hand, we have

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 3, 3, 1\} \tag{2}$$

  that is the order of elements and repetition of elements do not matter.

- The second is to define a set to be all objects satisfying one or more properties:
  - Let $A = \{$All the real numbers that satisfy $|x - 1| < 3\}$. Then

$$1 \in B, \qquad 4 \notin B. \tag{3}$$

  - Let $B = \{$All natural numbers n such that there are integers solving $x^n + y^n = z^n\}$. Thanks to Andrew Wiles we now know that

$$1, 2 \in C, \qquad 3, 4, \ldots \notin C. \tag{4}$$

  - Let $C = \{x \in B: x^2 = 9\}$. Then $3 \in D$, $-3 \notin D$.

  In general, a set defined this way is written as

$$A = \{x |\, P(x)\} \tag{5}$$

  which reads "A is the set of all x such that the statement $P(x)$ is true".

  Exercise 2. Write the above sets A, B, C in the form $\{x |\, P(x)\}$. Make $P(x)$ explicit.

  As seen above in the definition of C, we also use the notation

$$A = \{x \in B |\, P(x)\} \tag{6}$$

  to mean "A is the set of all x in B such that $P(x)$ is true." This is useful when the set B is previously defined and most of the x's under consideration are elements of B. For example, in real analysis we often write definitions like $A = \{x \in \mathbb{R} |\, \ldots.\}$.

  Exercise 3. Sometimes a set is defined as

$$A = \{f(x) |\, P(x)\} \tag{7}$$

  where f is a previously defined function. Explain what this means.

  Remark 2. In mathematics sets usually have infinitely many members. For such sets we have to take the second way. However, see the next remark.

Remark 3. The above two methods of writing a set are based on the following two assumptions:

1. Axiom of extensionality: A set is determined by its elements, that is, two sets with same elements are the same set.

2. Axiom of intentionality: A set can be determined by one or more properties. That is we can write

$$A = \{x|\, P(x) \text{ is true}\} \text{ or simply } A = \{x|\, P(x)\} \tag{8}$$

where $P(x)$ is a mathematical statement involving x.

These two axioms seem naturally true but unfortunately they lead to paradoxes. Fortunately, at the level of analysis, such paradoxes have little effect.

Exercise 4. (Russel's paradox) Let $A = \{S|\, S \notin S\}$. Show why this is a paradox.

Example 4. (Empty set) There is a special set called "empty set", denoted $\varnothing$, which is defined as a set with no element. In other words, there is no object a satisfying $a \in \varnothing$, or equivalently, every object a satisfies $a \notin \varnothing$.

Exercise 5. Prove that there is exactly one "empty set". Which axiom are you using in your argument?

Example 5. There are several special sets of numbers that are so important that they have special letters assigned to them.

- The set of natural numbers is denoted $\mathbb{N}$; So $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- The set of integers is denoted $\mathbb{Z}$; So $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$.
- The set of rational numbers is denoted $\mathbb{Q}$;
- The set of real numbers is denoted $\mathbb{R}$;
- The set of complex numbers is denoted $\mathbb{C}$.

Note that in the following of this course we will use these standard symbols.

1.1. Relations between sets.

Two sets A, B can have the following possible relations:

- Subset. If every element of A is also an element of B, we say A is a subset of B, denoted $A \subseteq B$. Using the logical statements in the last subsection, $A \subseteq B$ is defined as

$$x \in A \Longrightarrow x \in B. \tag{9}$$

  For example $\mathbb{N} \subseteq \mathbb{Z}$.

- Equal. If A, B have exactly the same elements, we say $A = B$. For example, if $A := \{x \in \mathbb{R}: x^2 + 1 = 0\}$, then $A = \varnothing$. On the other hand, if $B := \{x \in \mathbb{C}: x^2 + 1 = 0\}$, then $B = \{i, -i\}$.

  $A = B$ can be defined as

$$x \in A \Longleftrightarrow x \in B \tag{10}$$

  which immediately gives

$$A = B \text{ is equivalent to } [(A \subseteq B) \text{ and } (B \subseteq A)]. \tag{11}$$

  When the relation between A, B are not obvious, showing $A \subseteq B$ and then $B \subseteq A$ is most likely the only way to prove $A = B$.

- Proper subset. From the above definitions we know that if $A = B$, then $A \subseteq B$. However, when we talk about subsets, often we do not mean this trivial situation. Thus we define "A is a proper subset of B", denoted $A \subset B$, if $A \subseteq B$ but $A \neq B$. Sometimes it is also denoted as $A \subsetneq B$ to emphasize these.

How to prove set relations:

- $A \subseteq B$. To prove $A \subseteq B$, we show every $x \in A$ also $\in B$. More specifically, we show that an element $x \in A$, taken arbitrarily, must also be in B.

  Example 6. Let $A = \{x > 0\}$, $B = \{x > -1\}$. Prove $A \subseteq B$.

Proof. Let $x \in A$. Then $x > 0$. This gives $x > -1$ which means $x \in B$. Thus any $x \in A$ also $\in B$ which means $A \subseteq B$. □

- $A = B$. To show this we need to show $A \subseteq B$ and $B \subseteq A$.

  Example 7. Let $A = \{x | \, e^x > 1\}$ and $B = \{x > 0\}$, prove $A = B$.

  Proof. We first show $A \subseteq B$ and then $B \subseteq A$.

  - $A \subseteq B$. Take arbitrary $x \in A$, we have $e^x > 1 = e^0$. This gives $x > 0$ so $x \in B$.
  - $B \subseteq A$. Take arbitrary $x \in B$, we have $e^x > e^0 = 1$. This gives $e^x > 1$ so $x \in A$. □

- $A \subsetneq B$. Show $A \subseteq B$ and $A \neq B$. More specifically, first prove "any $x \in A$ also $\in B$", then prove "there is $y \in B$ such that $y \notin A$". This is usually done through explicitly finding this element y.

  Example 8. Let $A = \{x > 0\}$, $B = \{x > -1\}$. Prove $A \subsetneq B$.

  Proof. We have already proved $A \subseteq B$, therefore all we need to show is $A \neq B$. To show this we need to

  - either find an element of A which is not an element of B,
  - or find an element of B which is not an element of A.

  Since $A \subseteq B$, all elements of A are elements of B, so we try to find an element of B which is not an element of A. This is easy: $0 \in B$ but $0 \notin A$. □

Proposition 9. *We have*

- $\varnothing \subseteq A$ *for any set* A.
- *Let* A *be a set. If* $A \subseteq \varnothing$, *then* $A = \varnothing$.
- $A \subseteq B$ *and* $B \subseteq C$ *then* $A \subseteq C$.

Proof.

- *This is true because there is no element in* $\varnothing$.[1]
- *We need to show that* A *has no element. Assume the contrary, let* $x \in A$. *Then* $x \notin \varnothing$. *Therefore* $A \not\subseteq \phi$, *contradiction.*
- *Take arbitrary* $x \in A$. *As* $A \subseteq B$, $x \in B$. *As* $B \subseteq C$, $x \in C$. *Thus we have: for any* $x \in A$, $x \in C$ *which is exactly* $A \subseteq C$. □

1.2. Operations that create new sets.

Given two sets A, B, the following operations create new sets from A and B.

- Union. The union of two sets A, B is the new set obtained from putting all elements of A and all elements of B together. We denote

$$A \cup B := \{x | \, (x \in A) \vee (x \in B)\}. \tag{12}$$

- Intersection. The intersection of A, B is the set of all common elements of A, B:

$$A \cap B := \{x | \, (x \in A) \wedge (x \in B)\}. \tag{13}$$

- Set difference. The "set difference of B from A" is the set of all elements that are in A but not in B:

$$A - B := \{x | \, (x \in A) \wedge (x \notin B)\}. \tag{14}$$

1. *We need to show* $\{x \in \varnothing\} \Longrightarrow \{x \in A\}$. *But* $x \in \varnothing$ *is always false so the whole statement is always true no matter what* A *is.*

It is also called the "complement of B relative to A". In some textbooks the notation $A\backslash B$ is also used.

- Complement. Often all the sets relevent to our discussion are subsets of an "ambient set" X. For example in 314 almost all the sets we discuss are subsets of the ambient set $\mathbb{R}$. In this case $X-B$ is often denoted as B^c and called "complement of B".

Exercise 6. Prove the following: Let A, B be any set.

$$A-A=\varnothing; \quad A-\varnothing=A; \quad \varnothing-A=\varnothing; \tag{15}$$

$$A-B=A-(A\cap B); \tag{16}$$

$$A^c\cap B^c=(A\cup B)^c; \quad A^c\cup B^c=(A\cap B)^c. \tag{17}$$

Proposition 10. *The following are very useful in proving set relations.*

a) $A\subseteq C, B\subseteq C\Longrightarrow A\cup B\subseteq C$*;*

b) $C\subseteq A, C\subseteq B\Longrightarrow C\subseteq A\cap B$*;*

c) $A\subseteq B\Longrightarrow A\cap C\subseteq B\cap C$*;*

d) $A\subseteq B\Longrightarrow A\cup C\subseteq B\cup C$*.*

e) $A\subseteq B\Longrightarrow C-B\subseteq C-A$*.*

Proof.

a) We are given

$$x\in A\Longrightarrow x\in C \text{ and } x\in B\Longrightarrow x\in C \tag{18}$$

and need to show

$$(x\in A \text{ or } x\in B)\Longrightarrow x\in C. \tag{19}$$

We only need to deal with the case $x\in A$ or $x\in B$ is true. Thus at least one of $x\in A$,$x\in B$ is true. Say $x\in A$ is true. Then $x\in A\Longrightarrow x\in C$ gives $x\in C$ is true. Similarly if $x\in B$ is true we also conclude $x\in C$ is true. Consequently if $x\in A$ or $x\in B$ is true then $x\in C$ is true, which is exactly

$$(x\in A \text{ or } x\in B)\Longrightarrow x\in C. \tag{20}$$

b)–e) can be proved similarly and is omitted. □

Exercise 7. Prove b) – e).

Theorem 11. (Properties of set operations) *Let A, B, C be sets*

a) $A\cap B\subseteq A\subseteq A\cup B$*.*

b) $A\cup A=A; A\cap A=A$*.*

c) $A\cup B=B\cup A$*;* $A\cap B=B\cap A$*.*

d) $(A\cup B)\cup C=A\cup(B\cup C)$*;* $(A\cap B)\cap C=A\cap(B\cap C)$*.*

e) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$*;* $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$*.*

Proof. The proofs are quite similar, thus we will not give all the details but only prove the first half of e).

Recall that to prove $A=B$, all we need are $A\subseteq B$ and $B\subseteq A$.

- $[(A\cap B)\cup(A\cap C)]\subseteq A\cap(B\cup C)$. Since $B\subseteq B\cup C$, application of Proposition 10 gives $A\cap B\subseteq A\cap(B\cup C)$. Similarly we have $A\cap C\subseteq A\cap(B\cup C)$. Applying Proposition 10 again, we have

$$[(A\cap B)\cup(A\cap C)]\subseteq A\cap(B\cup C). \tag{21}$$

- $A\cap(B\cup C)\subseteq[(A\cap B)\cup(A\cap C)]$. This is a bit tricky. First consider the case $A\cap(B\cup C)=\varnothing$. In this case $A\cap(B\cup C)\subseteq$ any other set, so the claim holds.

  Otherwise, for every $x\in A\cap(B\cup C)$, by definition $x\in A$ and $x\in B$ or C. We discuss the two cases separately. If $x\in B$, then $x\in A\cap B\subseteq(A\cap B)\cup(A\cap C)$; If $x\in C$, then $x\in A\cap C\subseteq(A\cap B)\cup(A\cap C)$. □

Exercise 8. Prove the remaining claims of the theorem.

Remark 12. From property d) in the above theorem we see that the intersection of three sets: $A\cap B\cap C$ is well-defined.

Remark 13. A good tool to understand relations of sets is the Venn graph, or Venn diagram (the wiki page is a good enough reference). However, drawing a Venn graph is **NOT A PROOF** of set relations. It is just a help of visualizing what is going on and may inspire how the proof could be constructed.

1.3. Operations on arbitrary number of sets.

The operations $\cup$, $\cap$ can be generalized naturally to involve more than two or even infinitely many sets. For example we can define

$$A\cap B\cap C=\{x: x\in A \text{ and } x\in B \text{ and } x\in C\}. \tag{22}$$

More specifically, let $\mathcal{E}=\{E_\alpha\}_{\alpha\in A}$ be a collection of sets. Then

i. The union of $\mathcal{E}$ is

$$\cup_{\alpha\in A}E_\alpha:=\{x: x\in E_\alpha \text{ for at least one } \alpha\in A\}; \tag{23}$$

ii. The intersection of $\mathcal{E}$ is

$$\cap_{\alpha\in A}E_\alpha:=\{x: x\in E_\alpha \text{ for all } \alpha\in A\}. \tag{24}$$

Example 14. Let $E_a=\{x\in\mathbb{R}: x<1/a\}$ for $a\in A=\{a\in\mathbb{R}: a>1\}$. Calculate (meaning: give the simplest description possible, preferably one single formula) $\cap_{a\in A}E_a$ and $\cup_{a\in A}E_a$.
Solution.

- $\cap_{a\in A}E_a$. Solving such problems usually involve three steps.
  1. Determine the answer. If $a>1$, then $0<1/a$. So $\{x: x\leqslant 0\}\subseteq E_a$ for all a. As a gets larger, $1/a$ gets smaller so we are pretty sure $E:=\{x: x\leqslant 0\}$ should be the answer. What we need to do now is to show $\cap_{a\in A}E_a=E$. Remember our only method of showing equality of two sets?
  2. $\cap_{a\in A}E_a\subseteq E$. We need to show that if $x\in\mathbb{R}$ satisfies $x<1/a$ for all $a>1$, then $x\leqslant 0$. We prove by contradiction. Assume there is $x>0$ such that $x<1/a$ for all $a>1$. Set $a=\frac{1+x}{x}>1$. Then $1/a=\frac{x}{1+x}<x$ as $x>0$, contradiction.[2]
  3. $E\subseteq\cap_{a\in A}E_a$. We need to show that every $x\leqslant 0$ satisfies $x<1/a$ for every $a>1$. This is obvious as $x\leqslant 0<1/a$.
- $\cup_{a\in A}E_a$. The procedure is similar, we get $\cup_{a\in A}E_a=\{x: x<1\}$.

2. Think: Why not just set $a=1/x$? Then $1/a=x\geqslant x$, contradiction!

2. Sets of real numbers.

2.1. Subsets of $\mathbb{R}$.

The most important sets to us are subsets of the set of real numbers, that is $E \subseteq \mathbb{R}$.

Intervals.

One special class of subsets of $\mathbb{R}$ is intervals:

Definition 15. *Let a, b be real numbers. A closed interval is a set of the form*

$$[a, b] := \{x \in \mathbb{R} \colon a \leqslant x \leqslant b\}, \qquad [a, \infty) := \{x \in \mathbb{R} \colon a \leqslant x\} \tag{25}$$

$$(-\infty, b] := \{x \in \mathbb{R} \colon x \leqslant b\}; \qquad (-\infty, \infty) := \mathbb{R}. \tag{26}$$

An open interval is a set of the form

$$(a, b) := \{x \in \mathbb{R} \colon a < x < b\}, \qquad (a, \infty) := \{x \in \mathbb{R} \colon a < x\} \tag{27}$$

$$(-\infty, b) := \{x \in \mathbb{R} \colon x < b\}; \qquad (-\infty, \infty) := \mathbb{R}. \tag{28}$$

One can also define half-open, half-closed intervals:

$$[a, b) := \{x \in \mathbb{R} \colon a \leqslant x < b\}, \qquad (a, b] := \{x \in \mathbb{R} \colon a < x \leqslant b\}. \tag{29}$$

Remark 16. Note that $\mathbb{R}$ is both an open interval and a closed interval.

Example 17. Write the following in interval notation:

a) $A := \{x \in \mathbb{R} \colon |x - 3| \leqslant 1\}$;

b) $B := \{x \in \mathbb{R} \colon |x - 3| > 5\}$.

c) $(1, 2)^c$.

Solution.

a) For A we have $A = \{x \in \mathbb{R} \colon 2 \leqslant x \leqslant 4\}$ so $A = [2, 4]$;

b) $B = \{x \in \mathbb{R} \colon x > 8 \text{ or } x < -2\}$ so $B = (-\infty, -2) \cup (8, \infty)$.

c) $(1, 2) = \{x \in \mathbb{R} \colon 1 < x < 2\}$ so $(1, 2)^c = \{x \in \mathbb{R} \colon x \leqslant 1 \text{ or } x \geqslant 2\} = \{x \in \mathbb{R} \colon x \leqslant 1\} \cup \{x \in \mathbb{R} \colon x \geqslant 2\}$ which equals $(-\infty, 1] \cup [2, \infty)$.

Exercise 9. Prove the following:

a) $(a, b) \subseteq (c, d) \Longleftrightarrow [(a \geqslant c) \wedge (b \leqslant d)]$;

b) $(a, b) \subseteq [c, d] \Longleftrightarrow [(a \geqslant c) \wedge (b \leqslant d)]$;

c) $[a, b] \subseteq [c, d] \Longleftrightarrow [(a \geqslant c) \wedge (b \leqslant d)]$;

d) $[a, b] \subseteq (c, d) \Longleftrightarrow [(a > c) \wedge (b < d)]$.

Remember: To prove $\Longleftrightarrow$ you need to prove both $\Longrightarrow$ and $\Longleftarrow$!

Open sets and closed sets.

Using intervals we can define open and closed sets, which are crucial in real analysis.

Definition 18. *A set $E \subseteq \mathbb{R}$ is* open *if for every $x \in E$, there is an open interval $(a, b) \subseteq E$ such that $x \in (a, b)$. A set $E \subseteq \mathbb{R}$ is* closed *if its complement $E^c := \mathbb{R} - E$ is open.*

Remark 19. Traditionally, we say $\mathbb{R}$ and $\varnothing$ are both open and closed.

Lemma 20. *Open intervals are open, closed intervals are closed. Half-open, half-closed intervals are neither open nor closed.*

Proof. Let I be an open interval. For every $x \in I$, we have $x \in I \subseteq I$. Therefore I is open.

Next let $a, b \in \mathbb{R}$ and $[a, b]$ be an closed interval. Then we have $[a, b]^c = (-\infty, a) \cup (b, \infty)$. Take any $x \in [a, b]^c$. Then there are two cases:

- $x \in (-\infty, a)$. Since $(-\infty, a)$ is an open interval, we have $x \in (-\infty, a) \subseteq [a, b]^c$.
- $x \in (b, \infty)$. Similarly, (b, ∞) is an open interval so we have $x \in (b, \infty) \subseteq [a, b]^c$.

The other three cases are easier:

- $[a, \infty)$: We have $[a, \infty)^c = (-\infty, b)$ open;
- $(-\infty, b]$: We have $(-\infty, b]^c = (b, \infty)$ open;
- $(-\infty, \infty)$: We have $(-\infty, \infty)^c = \varnothing$ open.

Finally we show that $[a, b)$ and $(a, b]$ are neither open nor closed. For $[a, b)$, to see that it is not open, take $x = a \in [a, b)$. Then for every open interval (c, d) containing $x = a$, we have $c < a$ and therefore $\frac{c+a}{2} \in (c, d)$ but $\frac{c+a}{2} \notin [a, b)$. To see that it is not closed, we consider $[a, b)^c = (-\infty, a) \cup [b, \infty)$. Take $x = b$ and argue similarly, we see that $[a, b)^c$ is not open. Therefore $[a, b)$ is neither open nor closed. The proof for $(a, b]$ is similar. □

Exercise 10. Prove by definition:

a) $(0, 1) \cup (1, 2)$ is open;

b) $[1, 2] \cup [3, 4]$ is closed.

Lemma 21. *If E is open, then E^c is closed; If E is closed then E^c is open.*

Proof. The second part is by definion. For the first part, because $(E^c)^c = E$, E^c is closed if E is open. □

Theorem 22. *We have the following results about intersection and union of sets:*

a) The intersection of finitely many open sets is open; The union of open sets is open.

b) The intersection of closed sets is closed; The union of finitely many closed sets is closed.

Proof. We prove a). b) follows from a), Lemma 21, and De Morgan's rule of set operations:

$$(\cap_{\alpha \in A} E_\alpha)^c = \cup_{\alpha \in A} E_\alpha^c; \qquad (\cup_{\alpha \in A} E_\alpha)^c = \cap_{\alpha \in A} E_\alpha^c. \tag{30}$$

- Intersection of finitely many open sets. Denote these sets by $E_1, \ldots, E_n$. We show that for every $x \in \cap_{i=1}^n E_i$, there is $(a, b) \subseteq \cap_{i=1}^n E_i$ such that $x \in (a, b)$.

  As E_1 is open, there is $(a_1, b_1) \subseteq E_1$ with $x \in (a_1, b_1)$;

  As E_2 is open, there is $(a_2, b_2) \subseteq E_2$ with $x \in (a_2, b_2)$;

  Doing this for all E_i, we obtain $(a_i, b_i) \subseteq E_i$ such that $x \in (a_i, b_i)$.

  Now set $a = \max\{a_1, \ldots, a_n\}$ and $b = \min\{b_1, \ldots, b_n\}$. We claim that $a < x < b$. Since $x \in (a_i, b_i)$, we have $a_i < x < b_i$ for all $i = 1, \ldots, n$. Therefore $a = \max\{a_1, \ldots, a_n\} < x < \min\{b_1, \ldots, b_n\} = b$. Thus we have (a, b) is an open interval and $x \in (a, b)$. Finally, as $a \geqslant a_i$, $b \leqslant b_i$, we have $(a, b) \subseteq (a_i, b_i) \subseteq E_i$ for all $i = 1, 2, \ldots, n$. Therefore $(a, b) \subseteq \cap_{i=1}^n E_i$.

- Union of (could be infinitely many) open sets. Denote these sets by E_α with $\alpha \in A$ an index set. Take any $x \in \cup_{\alpha \in A} E_\alpha$. By definition there is E_{α_0} such that $x \in E_{\alpha_0}$. Since E_{α_0} is open, there is $a, b \in \mathbb{R}$ such that $x \in (a, b) \subseteq E_{\alpha_0} \subseteq \cup_{\alpha \in A} E_\alpha$. □

Remark 23. Note that the "union of open sets" and "intersection of closed set" in the above theorem can in principle involve infinitely many sets. For example $(1/x, \infty)$ is open for $x > 0$. Then we know $\cup_{x>0}(1/x, \infty)$ is open. On the other hand, the intersection of infinitely many open sets may be closed, for example $\cap_{n \in \mathbb{N}}(-1/n, 1/n)$; The union of infinitely many closed sets may be open, for example $\cup_{n \in \mathbb{N}}\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. Of course it may also be half-open-half-closed.

Remark 24. From the above theorem we see why it is a good idea to say $\mathbb{R}$ and $\varnothing$ are both open and closed.

Theorem 25. (Structure of open sets) *Let $E \subseteq \mathbb{R}$ be open. Then there are $a_i, b_i \in \mathbb{R}$, $i \in \mathbb{N}$ such that $E = \cup_{i \in \mathbb{N}}(a_i, b_i)$.*

Proof. The proof is beyond the level of this course and is omitted. □

2.2. Properties of sets of real numbers.

Sup and Inf.

For a set of finitely many real numbers, we often talk about its "largest" and "smallest" elements: its maximum and minimum.

Definition 26. (max and min) *Let A be a nonempty set of numbers. Then the maximum of A is an element $x \in A$ such that*

$$\forall y \in A \qquad y \leqslant x. \tag{31}$$

Similarly, the minimum of A is an element $z \in A$ such that

$$\forall y \in A \qquad y \geqslant z. \tag{32}$$

The maximum and minimum of a set give us a rough idea of how "spread out" the set is. However, when the set is infinite, maximum or minimum may not exist.

Example 27. (max/min may not exist) Let $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\min A = 0$, while $\max A$ does not exist.

- $\min A = 0$. Checking definition of min we see that we need to prove
  1. $0 \in A$; This is true as $0 = 1 - \frac{1}{1} \in A$.
  2. For any $x \in A$, $x \geqslant 0$. Let $x \in A$ be arbitrary. Then there is $n \in \mathbb{N}$ such that $x = 1 - \frac{1}{n} \geqslant 0$.

  Thus we have proved $\min A = 0$.
- $\max A$ does not exist. Assume the contrary, then there is $a_{\max} \in A$. Then there is $n_0 \in \mathbb{N}$ such that $a_{\max} = 1 - \frac{1}{n_0}$. Taking $n > n_0$ we have $a_{\max} < 1 - \frac{1}{n} \in A$, contradiction.

Exercise 11. Let $S \subset \mathbb{R}$ be a finite set. Prove that $\max S$ and $\min S$ exist.

Exercise 12. Give an example of a infinite set $A \subset \mathbb{R}$ whose maximum and minimum both exist. Justify your answer.

To fix this situation, we introduce the notion of supreme and infimum, two numbers characterize a how spread out a set of (possibly infinite) real numbers is.

Definition 28. (sup and inf) *Let A be a nonempty set of numbers. The supreme of A is defined as*

$$\sup A = \min \{b \in \mathbb{R} : b \geqslant a \text{ for every } a \in A\}. \tag{33}$$

If $\{b \in \mathbb{R} : b \geqslant a \text{ for every } a \in A\} = \varnothing$, we write

$$\sup A = \infty; \tag{34}$$

The infimum of A is defined as

$$\inf A = \max \{b \in \mathbb{R} : b \leqslant a \text{ for every } a \in A\}. \tag{35}$$

If $\{b \in \mathbb{R} : b \geqslant a \text{ for every } a \in A\} = \varnothing$, write

$$\inf A = -\infty. \tag{36}$$

Example 29. (max/min may not exist) Let $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\sup A = 1$, $\inf A = \min A = 0$, while $\max A$ does not exist.

- $\sup A = 1$. We show two things:
  1. $\forall a \in A, 1 \geqslant a$. Take any $a \in A$. Then there is $n \in \mathbb{N}$ such that $a = 1 - \frac{1}{n} < 1$.
  2. $\forall b \in \mathbb{R}$ such that $b \geqslant a$ for all $a \in A$, $b \geqslant 1$. Since $b \geqslant a$ for all $a \in A$, $b \geqslant 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Assume $b < 1$. Taking $n > \frac{1}{1-b}$ leads to contradiction.
- $\inf A, \min A = 0$. Omitted.
- $\max A$ does not exist. Assume the contrary, then there is $a_{\max} \in A$. Then there is $n_0 \in \mathbb{N}$ such that $a_{\max} = 1 - \frac{1}{n_0}$. Taking $n > n_0$ we have $a_{\max} < 1 - \frac{1}{n} \in A$, contradiction.

Definition 30. (Upper/lower bound) *Let $A \subseteq \mathbb{R}$ be nonempty. $a \in \mathbb{R}$ is said to be a upper bound of A if and only if*

$$\forall x \in A \qquad x \leqslant a; \tag{37}$$

$b \in \mathbb{R}$ is said to be a lower bound of A if and only if

$$\forall x \in A \qquad x \geqslant b. \tag{38}$$

Remark 31. Thus $\sup A$ is "least upper bound" of A. Similarly, $\inf A$ is "greatest lower bound" of A.

Remark 32. The advantage of supreme and infimum is that they always exist.

Exercise 13. Try to prove the above claim. Do you encounter any difficulty?

Remark 33. In fact, the existence of sup and inf is part of the definition of the set of real numbers $\mathbb{R}$.

Exercise 14. More precisely, in the definition of $\mathbb{R}$ we only need to put in existence of one of the two. Prove

a) If $\sup A$ exists for any $A \subseteq \mathbb{R}$, then $\inf B$ exists for any $B \subseteq \mathbb{R}$.

b) If $\inf A$ exists for any $A \subseteq \mathbb{R}$, then $\sup B$ exists for any $B \subseteq \mathbb{R}$.

sup and inf are generalizations of max and min.

Proposition 34. *Let $A \subseteq \mathbb{R}$. If* $\max A$ *exists, then* $\sup A = \max A$*; Similarly, if* $\min A$ *exists, then* $\inf A = \min A$.

Proof. Let $a_{\max} = \max A$. Set $B = \{b \in \mathbb{R}: b \geqslant a \text{ for every } a \in A\}$. We need to show that $a_{\max} = \min B$, that is

1. $a_{\max} \in B$. As $a_{\max} = \max A$, we have $a_{\max} \geqslant a$ for all $a \in A$. Therefore $a_1 \in B$;
2. $\forall b \in B$, $a_{\max} \leqslant b$. Take any $b \in B$. Then $b \geqslant a$ for all $a \in A$. In particular $b \geqslant a_{\max}$.

The proof for the inf/min part is similar. □

Exercise 15. Identify a real number $r \in \mathbb{R}$ with the set $M_r := \{x \in \mathbb{R} | \, x < r\}$.

a) Prove that the sets correspond to $\sup A$ and $\inf A$ are

$$\cup_{r \in A} M_r, \qquad \cap_{r \in A} M_r \qquad \text{respectively.} \tag{39}$$

b) What sets correspond to $r_1 \pm r_2$? Justify your answer.

c) What set corresponds to the product $r_1 r_2$? Justify your answer. (Be careful about sign!)

d) What set corresponds to $1/r$?

Theorem 35. **(Monotone property)** *Suppose $A \subseteq B$ are nonempty subsets of* $\mathbb{R}$. *Then*

a) $\sup B \geqslant \sup A$.

b) $\inf B \leqslant \inf A$.

Proof. We prove a) and leave b) as exercise.

If $\sup B = \infty$, then $\sup B \geqslant \sup A$ holds; If $\sup B \in \mathbb{R}$, then by definition we have

$$\sup B \geqslant b \text{ for every } b \in B \Longrightarrow \sup B \geqslant a \text{ for every } a \in A \tag{40}$$

because $A \subseteq B$. By definition of $\sup A$ we conclude that $\sup B \geqslant \sup A$. □

Exercise 16. Are there simple formulas of "sup" and "inf" for $A \cap B, A \cup B, A - B$? Justify your answers.

Nested sets.

Definition 36. **(Nested sets)** *A sequence of sets $\{I_n\}_{n \in \mathbb{N}}$ is said to be nested if*

$$I_1 \supseteq I_2 \supseteq \cdots \tag{41}$$

Theorem 37. **(Nested interval)** *If $I_n = [a_n, b_n]$ with $a_n, b_n \in \mathbb{R}$ is nested, then $\cap_{n=1}^{\infty} I_n$ is not empty.*

- **Discussion.**

  First understand what is the implication of $I_1 \supseteq I_2 \supseteq \cdots$. This means

  $$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant \cdots \leqslant b_n \leqslant \cdots \leqslant b_2 \leqslant b_1. \tag{42}$$

  Next to understand the proof, we think about the case where there are only finitely many intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n]$. Then clearly any $x \in [a_n, b_n]$ would belong to $\cap_{k=1}^{n} I_n$, for the following reason:

  $$a_n = \max\{a_1, \ldots, a_n\} \geqslant \text{ any } a_k; \quad b_n = \min\{b_1, \ldots, b_n\} \leqslant \text{ any } b_k. \tag{43}$$

Now return to the proof of the theorem. We have infinitely many intervals so it may not be possible to pick the largest a_n or the smallest b_n. But we always have $\sup a_n$ and $\inf b_n$.

Proof. Let $A=\{a_n\}$ and $B=\{b_n\}$. Consider $a=\sup A$ and $b=\inf B$. We prove

- $a \leqslant b$. Assume the contrary, that is $b<a$. Since $a=\sup A$, by definition b is not an upper bound of A, there is there is $n_0 \in \mathbb{N}$ such that $a_{n_0}>b$. As the intervals are nested, we have $a_{n_0} \leqslant b_n$ for all $n \in \mathbb{N}$ that is a_{n_0} is a lower bound of B. Now $b=\inf B$ by definition implies $b \geqslant a_{n_0}$. Contradiction.
- $[a,b] \subseteq \cap_{n=1}^{\infty} I_n$. We show that for any $n \in \mathbb{N}$, $[a,b] \subseteq [a_n, b_n]$. To show this we only need $a \geqslant a_n$ and $b \leqslant b_n$. Both follow directly from $a=\sup A$ and $b=\inf B$. □

Exercise 17.

a) Give a proof using $\lim a_n$ and $\lim b_n$ after we have discussed limits.

b) If furthermore $\lim_{n \longrightarrow \infty} b_n - a_n = 0$, show that $\cap_{n=1}^{\infty} I_n$ consists of a single point.

Remark 38.

- $a_n, b_n \in \mathbb{R}$ is necessary. Otherwise we can take $I_n = [a_n, \infty)$ with $a_n \longrightarrow \infty$ which leads to $\cap_{n=1}^{\infty} I_n \neq \varnothing$.
- It is also necessary that the intervals are closed. Counter-examples are $I_n = (0, 1/n)$, or $I_n = (0, 1/n]$.

Theorem 39. (Approximation of sup and inf) *Let $A \subseteq \mathbb{R}$ with $\sup A, \inf A \in \mathbb{R}$. Then for every $\varepsilon > 0$, there are $a, b \in A$ such that*

$$\sup A - a < \varepsilon; \qquad b - \inf A < \varepsilon \tag{44}$$

Proof. We prove the sup case and left the inf case as exercise. Assume the contrary. Then there is $\varepsilon_0 > 0$ such that for all $a \in A$, $\sup A - a \geqslant \varepsilon_0$. Now set $a_{\sup} := \sup A - \varepsilon_0/2$. We have $a_{\sup} > a$ for all $a \in A$ but $a_{\sup} < \sup A$. Contradiction. □

Remark 40. By setting $\varepsilon = 1/n$, we can obtain a sequence $\{x_n\}$ with $x_n \in A$ such that $x_n \longrightarrow \sup A$ (or $\inf A$). However note that x_n may not be different elements from one another. For example when A is finite, we basically will have to take the sequence $x_n = a_{\max}$.

Exercise 18. Can we define "sup", "inf" for sets in $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{N}$? Do "sup", "inf" always exist in subsets of these ambient sets?

2.3. Extended real numbers.

It is often convenient to add to $\mathbb{R}$ two elements $+\infty$, $-\infty$ to form the set of "extended real numbers". Most of the arithmetics on $\mathbb{R}$ can be carried over:

$$x + \infty = \infty, \qquad x - \infty = -\infty, \qquad x \in \mathbb{R} \tag{45}$$

$$x \cdot \infty = \infty, \qquad x \cdot (-\infty) = -\infty, \qquad x > 0 \tag{46}$$

$$x \cdot \infty = -\infty, \qquad x \cdot (-\infty) = \infty, \qquad x < 0 \tag{47}$$

$$\infty + \infty = \infty, \qquad -\infty - \infty = -\infty \tag{48}$$

$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty \quad \infty \cdot (-\infty) = (-\infty) \cdot (\infty) = -\infty \tag{49}$$

and the following are not involved: $\infty - \infty$ or $0 \cdot (+\infty)$ or $0 \cdot (-\infty)$.

3. Functions.

Functions used to be defined through formulas. However, as people gain more understanding of sciences and mathematics, it became necessary to study functions that cannot be described by formulas. The current understanding is to define functions through what they do: How they map the input value to the output value.

3.1. Definitions.

Definition 41. (Functions) *A function $f: A \mapsto B$ is a rule assigning to each element $a \in A$ exactly one element in B, this element is denoted $f(a)$. We call A the domain of the function and B the range of the function.*

Example 42. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$. Let the rule be given by

$$1 \longrightarrow a \tag{50}$$
$$2 \longrightarrow c \tag{51}$$
$$3 \longrightarrow b \tag{52}$$
$$4 \longrightarrow d \tag{53}$$
$$5 \longrightarrow d \tag{54}$$

then this is a function with domain A and range B.

On the other hand, the rule

$$1 \longrightarrow a \tag{55}$$
$$1 \longrightarrow b \tag{56}$$
$$2 \longrightarrow c \tag{57}$$
$$3 \longrightarrow e \tag{58}$$

fails to be a function because

- It does not assign an element in B to **every** $a \in A$;
- For some $a \in A$ it assigns more than one element in B.

Remark 43. It is important to keep in mind that a function is a triplet: Domain A, Range B, Rule f. Changing any one of the three leads to a different function. Rigorously speaking, $\sin x$ over $(-1, 1)$ and $\sin x$ over $(-2, 2)$ are two different function, they just happen to coincide over $(-1, 1)$. In real analysis this may seem like some annoying triviality, but this understanding is extremely important when studying Complex Analysis, Functional Analysis, and many other higher level analysis courses.

Definition 44. (Sequence) *A sequence is a function with domain $\mathbb{N}$.*

Definition 45. (Restriction of functions) *Let $f: A \mapsto B$ be a function. Let $C \subset A$. Then we can define a new function, called "restriction of f on C" by keeping the same rule but change the domain to C.*

Example 46. Consider the function $f: \mathbb{R} \mapsto \mathbb{R}$ defined through $x \mapsto \sin x$. Then its restriction on $\mathbb{N}$ is the sequence $\{\sin n\}$.

3.2. When functions meet sets.

Image and pre-image.

We can consider the effect of a function on subsets of either the domain or the range.

Definition 47. (Image) *Let A, B be sets and let $f: A \mapsto B$ be a function. The image of a subset $S \subseteq A$ under f is defined as*

$$f(S) := \{b \in B: \exists a \in S \text{ such that } f(a) = b\} \qquad \text{or simply } f(S) := \{f(a) | a \in S\} \tag{59}$$

The image of f is defined as the special case

$$\mathrm{Image}(f) := f(A). \tag{60}$$

Example 48. Let $A = B = \mathbb{R}$, $S = \{a \in \mathbb{R}: 0 < a < \pi\}$, $f = \sin$. Then we have

$$\mathrm{Image}(f) = \{x \in \mathbb{R}: -1 \leqslant x \leqslant 1\}. \tag{61}$$

$$f(S) = \{x \in \mathbb{R}: 0 < x \leqslant 1\}. \tag{62}$$

From this example we see that the image of f may only be a proper subset of B.

When studying functions, it is often important to study those $a \in A$ such that $f(a)$ has certain property. More precisely, we need a notation for those $a \in A$ such that $f(a)$ belongs to a certain subset $S \subseteq B$.

Definition 49. (Pre-image) *Let A, B be sets and let $f: A \mapsto B$ be a function. The pre-image of a subet $S \subseteq B$ is defined as*

$$f^{-1}(S) := \{a \in A | f(a) \in S\}. \tag{63}$$

Remark 50. When $S = \{s\}$ has exactly one element, $f^{-1}(S)$ is called a "level set" of the function f.

Example 51. Let $A = B = \mathbb{R}$, $f = \sin$, $S = \{1\}$. Then

$$f^{-1}(S) = 2k\pi + \frac{\pi}{2}, \qquad k \in \mathbb{Z}. \tag{64}$$

If we let $S = \{2\}$, then clearly

$$f^{-1}(S) = \varnothing. \tag{65}$$

We see that the pre-image of a single element may not be a single element, it may also be empty or contain more than one element.

Exercise 19. (Functions and Sets) Let X be an ambient space. For any $A \subseteq X$ define a function $f_A: X \mapsto \mathbb{R}$ by $f_A(x) = 1$ if $x \in A$ and 0 otherwise.

a) Prove that

$$f_{A \cap B} = f_A \cdot f_B. \tag{66}$$

b) Use a) to prove

$$A \cap (B \cap C) = (A \cap B) \cap C. \tag{67}$$

c) Find similar formulas for

$$f_{A \cup B}, \quad f_{A - B}, \quad f_{A^c}. \tag{68}$$

d) Use the above to prove other set relations.

Effects of functions on set relations.

Let $f: X \mapsto Y$ be a function. Let A, B be two subsets of the domain X. Then there are three possible relations:

$$A \subseteq B, \quad A = B, \quad A \subset B. \tag{69}$$

We can discuss what we can conclude for the relation between $f(A)$ and $f(B)$.

Similarly we can study what the effect of f^{-1} is on subsets $S, T \subseteq Y$, in each of the following situations:

$$S \subseteq T, \quad S = T, \quad S \subset T. \tag{70}$$

Lemma 52. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X$ and $S, T \subseteq Y$. Then the following holds.*

a) If $A \subseteq B$ then $f(A) \subseteq f(B)$.

b) If $S \subseteq T$ then $f^{-1}(S) \subseteq f^{-1}(T)$.

Proof. Left as exercise. □

Exercise 20. Prove Lemma 52.

Remark 53. Note that $A \subset B$ ($S \subset T$) does not imply $f(A) \subset f(B)$ ($f^{-1}(S) \subset f^{-1}(T)$). The construction of counterexamples are left as exercises.

Effects of functions on set operations.

Theorem 54. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X$ and $S, T \subseteq Y$. Then*

a) $f(A \cap B) \subseteq f(A) \cap f(B)$.

b) $f(A \cup B) = f(A) \cup f(B)$.

c) $f(A - B) \supseteq f(A) - f(B)$.

d) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

e) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

f) $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$.

Proof.

a) Since $A \cap B \subseteq A$, Lemma 52 gives $f(A \cap B) \subseteq f(A)$; Application of the same lemma to $A \cap B \subseteq B$ gives $f(A \cap B) \subseteq f(B)$. Therefore $f(A \cap B) \subseteq f(A) \cap f(B)$.

b) We need to show $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A \cup B)$.

- $f(A \cup B) \subseteq f(A) \cup f(B)$. Take any $y \in f(A \cup B)$. Then there is $x \in A \cup B$ such that $y = f(x)$. Now $x \in A \cup B$ has two cases: $x \in A$ and $x \in B$. In the first case we have $y \in f(A)$ and in the second we have $y \in f(B)$. Therefore $x \in A \cup B$ implies $y \in f(A) \cup f(B)$. So $f(A \cup B) \subseteq f(A) \cup f(B)$.
- $f(A) \cup f(B) \subseteq f(A \cup B)$. Since $A \subseteq A \cup B$, application of Lemma 52 gives $f(A) \subseteq f(A \cup B)$. Application of the same lemma to $B \subseteq A \cup B$ gives $f(B) \subseteq f(A \cup B)$. Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$.

c) Left as exercise.

d) We need to show $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

- $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$. As $S \cap T \subseteq S$, application of Lemma 52 gives $f^{-1}(S \cap T) \subseteq f^{-1}(S)$. The same lemma applied to $S \cap T \subseteq T$ gives$f^{-1}(S \cap T) \subseteq f^{-1}(T)$. Therefore $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

- $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$. Take any $x \in f^{-1}(S) \cap f^{-1}(T)$. As $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S)$, we have $x \in f^{-1}(S)$ so $f(x) \in S$. On the other hand, $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(T)$ gives $f(x) \in T$. Therefore $f(x) \in S \cap T$ which means $x \in f^{-1}(S \cap T)$. Thus ends the proof for $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

e) Left as exercise.

f) Left as exercise. □

Exercise 21. Prove c), e), f) of the above theorem.

3.3. Composite functions and inverse functions.

Composite function.

Definition 55. (Composite function) *Let $f: A \mapsto B$ and $g: C \mapsto D$. If $B \subseteq C$, we can define the composite function $g \circ f: A \mapsto D$ through*

$$\forall x \in A \qquad (g \circ f)(x) := g(f(x)). \tag{71}$$

Exercise 22. Let $f(x) = \sin x$, $g(x) = x^2$, show that $f \circ g \neq g \circ f$.

Exercise 23. Prove that if $f \circ (g \circ h)$ is well-defined, so is $(f \circ g) \circ h$ and furthermore $f \circ (g \circ h) = (f \circ g) \circ h$.

Inverse function.

Given a function $f: A \mapsto B$, it is often necessary to understand the rule relating $f(a)$ back to a, that is is we would like to find a function $g: B \mapsto A$ such that

$$g(f(a)) = a \text{ for every } a \in A; \qquad f(g(b)) = b \text{ for every } b \in B. \tag{72}$$

Such a function is called an inverse function of f. However as we have seen, for general function this is not possible. Now let's see what extra conditions we need.

Remark 56. Note that in this course f^{-1} always denotes the pre-image, not the inverse function. More precisely, given $f: A \mapsto B$, f^{-1} is a function from the set of subsets of B to the set of subsets of A, while the inverse function g, if it exists, is a function from B to A.

Definition 57. (one-to-one,onto,bijection) *Let A, B be sets and $f: A \mapsto B$ a function.*

We say f is one-to-one if whenever $f(a_1) = f(a_2)$, we have $a_1 = a_2$.

We say that f is onto if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

We say f is a bijection if it is both one-to-one and onto.

Example 58. Consider the following functions: $A = B = \mathbb{R}$,

$$f_1(x) = 2\,x + 4; \quad f_2(x) = \arctan x; \quad f_3(x) = \sin x; \quad f_4(x) = 2\,x^3 + x^2 + 12\,x + 4. \tag{73}$$

Then f_1 is one-to-one and onto, f_2 is one-to-one but not onto, f_3 is neither, f_4 is onto but not one-to-one. We give proof to the claim about f_2 and leave others as exercise.

Proof of f_2 being one-to-one but not onto.

To prove that f_2 is one-to-one, we need to show that whenever $f_2(a_1) = f_2(a_2)$, we must have $a_1 = a_2$. One way to show this is through Fundamental Theorem of Calculus:

$$f_2(a_2) - f_2(a_1) = \int_{a_1}^{a_2} f_2'(x)\,\mathrm{d}x = \int_{a_1}^{a_2} \frac{1}{1 + x^2}\,\mathrm{d}x. \tag{74}$$

Now if $a_2 > a_1$, from the above we have $f_2(a_2) > f_2(a_1)$; If $a_2 < a_1$, we have $f_2(a_2) < f_2\,(a_1)$. Therefore if $f_2(a_1) = f_2(a_2)$, we must have $a_1 = a_2$.

To show that f_2 is not onto, all we need is a counter-example. That is all we need is one $b\in B=\mathbb{R}$ such that there is no $a\in A=\mathbb{R}$ such that $f_2(a)=b$. This is easy. For example $b=3$.

Note 59. Is the following “proof” of f_2 being one-to-one correct?

If $\arctan a_1=\arctan a_2$, taking $\tan$ gives $a_1=a_2$.

Theorem 60. *f has an inverse function if and only if f is a bijection.*

Proof. We need to prove

$$f \text{ has an inverse function} \Longleftrightarrow f \text{ is a bijection.} \tag{75}$$

Recall that we need to prove $\Longrightarrow$ and $\Longleftarrow$.

- $\Longrightarrow$. Let g be an inverse function of f. As for every $b\in B$, $f(g(b))=b$, f is onto. Now we show f is one-to-one. Let $f(a_1)=f(a_2)$. As g is a function, it maps $f(a_1)=f(a_2)$ to a single element in a:
$$a_1=g(f(a_1))=g(f(a_2))=a_2. \tag{76}$$
- $\Longleftarrow$. Since f is a bijection, for every $b\in B$ there is a unique $a\in A$ such that $f(a)=b$. We define
$$g(b)=a. \tag{77}$$
Thus automatically $g(f(a))=a$. On the other hand, $f(g(b))=f(a)=b$. So g is an inverse function. □

Exercise 24. Let $f: A\mapsto B$ be a function.

a) Assume there is a function $g: B\mapsto A$ such that $(g\circ f)(x)=x$ for all $x\in A$. Prove that f has an inverse function and furthermore it is exactly g.[3]

b) Formulate a similar claim assuming existence of g with $(f\circ g)(y)=y$ for all $y\in B$, and prove your claim.

3.4. Functions with range $\mathbb{R}$.

For functions with range $\mathbb{R}$, that is for functions taking real values, we can talk about how “spread out” the image is through the notion of supreme and infimum of sets of real numbers.

Definition 61. (Sup and Inf of functions) *Let $f: E\mapsto\mathbb{R}$ be a function. We define*

$$\sup_{x\in E} f(x):=\sup f(E); \qquad \inf_{x\in E} f(x):=\inf f(E). \tag{78}$$

Example 62. $\sup_{x\in(-1,1)}x^2=1$, $\inf_{x\in(-1,1)}x^2=0$.

Theorem 63. (sup and inf under operations of functions) *Let f,g be functions with domains containing $E\subseteq\mathbb{R}$. Let $c\in\mathbb{R}$ be a* positive *number. Then*

a) $\sup_{x\in E}(c\,f)=c\sup_{x\in E}f$; $\inf_{x\in E}(c\,f)=c\inf_{x\in E}f$;

b) $\sup_{x\in E}(-f)=-\inf_{x\in E}f$; $\inf_{x\in E}(-f)=-\sup_{x\in E}f$;

c) $\sup_{x\in E}(f+g)\leqslant\sup_{x\in E}f+\sup_{x\in E}g$;

d) $\inf_{x\in E}(f+g)\geqslant\inf_{x\in E}f+\inf_{x\in E}g$.

The above holds even when the supreme/infimum is ∞ or $-\infty$.

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3. This is an example of the following fact in group theory: if a has a left (or right) inverse, then it has an inverse.

Proof. We only prove b), c) here. Other cases are left as exercise.

- b). We show $\sup_{x\in E}(-f) = -\inf_{x\in E} f$. Once this is done setting $g = -f$ gives the other half. Let $a = \inf_{x\in E} f$ (maybe $-\infty$). We need to show

  1. $-a \geqslant -f(x)$ for all $x \in E$. Since $a = \inf_{x\in E} f$, by definition $a \leqslant f(x)$ for all $x \in E$. Therefore $-a \geqslant -f(x)$ for all $x \in E$.
  2. For any $b \in \mathbb{R}$ satisfying $b \geqslant -f(x)$ for all $x \in E$, we have $b \geqslant -a$. Since $b \geqslant -f(x)$ for all $x \in E$, we have $-b \leqslant f(x)$ for all $x \in E$. Since $a = \inf_{x\in E} f$, $a \geqslant -b$. Therefore $b \geqslant -a$.

  Note that the above argument still holds when $a = -\infty$.

- c). Denote $a = \sup_{x\in E} f$, $b = \sup_{x\in E} g$, we need to show that $a + b \geqslant f(x) + g(x)$ for all $x \in E$. If one of a, b is ∞, then we have $a + b = \infty \geqslant f(x) + g(x)$ for all $x \in E$.[4] If both a, $b \in \mathbb{R}$, take any $x \in E$. We have $a = \sup_{x\in E} f \geqslant f(x)$ and $b = \sup_{x\in E} g \geqslant g(x)$. Consequently $a + b \geqslant f(x) + g(x)$. □

Exercise 25. Prove a), d).

Example 64. The inequalities in c),d) may hold strictly. The reason is that the "peak" of f and the "peak" of g may not be at the same location. For example, take $f = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ and $g = \begin{cases} 1 & 1 \leqslant x < 2 \\ 0 & \text{elsewhere} \end{cases}$, we have $\sup_{x\in\mathbb{R}} f = \sup_{x\in\mathbb{R}} g = 1$, but $\sup_{x\in\mathbb{R}}(f + g) = 1 < 1 + 1 = 2$.

Remark 65. For more on functions, see [Sib09] Chapters 3, 4.

3.5. Important functions.

Absolute value.

Definition 66. (Absolute value function) *The function* $|\cdot|: \mathbb{R} \mapsto \mathbb{R}$ *is defined through*

$$|x| := \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}. \tag{79}$$

Exercise 26. Let $x, y \in \mathbb{R}$. Prove the following

a) $|x| = 0 \Longleftrightarrow x = 0$;

b) $|x\,y| = |x|\,|y|$;

c) $||x| - |y|| \leqslant |x + y| \leqslant |x| + |y|$.

Exercise 27. Let $x_1, \ldots, x_n \in \mathbb{R}$. Prove

$$|x_1 + \cdots + x_n| \leqslant |x_1| + \cdots |x_n|. \tag{80}$$

Max and Min.

Exercise 28. Let $x \in \mathbb{R}$. Prove that

$$|x| = \max\{x, -x\}. \tag{81}$$

Exercise 29. Let $x, y \in \mathbb{R}$. Prove that

$$\max\{x, y\} = \frac{x + y}{2} + \frac{|x - y|}{2}; \qquad \min\{x, y\} = \frac{x + y}{2} - \frac{|x - y|}{2}. \tag{82}$$

Odd and Even functions.

4. Note that by definition sup can only be real number or ∞, while inf can only be real number or $-\infty$.

Definition 67. *A function* $f: E \subseteq \mathbb{R} \mapsto \mathbb{R}$ *is odd if and only if*

$$\forall x \in E \qquad f(x) = -f(-x) \tag{83}$$

It is even if and only if

$$\forall x \in E \qquad f(x) = f(-x). \tag{84}$$

Remark 68. From the above it is clear that E cannot be arbitrary, it has to satisfy

$$(x \in E) \Longrightarrow (-x \in E). \tag{85}$$

Monotone functions.

Definition 69. *A function* $f: E \subseteq \mathbb{R} \mapsto \mathbb{R}$ *is*

- *increasing if and only if*
$$x, y \in E, \ x < y \Longrightarrow f(x) \leqslant f(y); \tag{86}$$
- *strictly increasing if and only if*
$$x, y \in E, \ x < y \Longrightarrow f(x) < f(y); \tag{87}$$
- *decreasing if and only if*
$$x, y \in E, \ x < y \Longrightarrow f(x) \geqslant f(y); \tag{88}$$
- *strictly decreasing if and only if*
$$x, y \in E, \ x < y \Longrightarrow f(x) > f(y); \tag{89}$$
- *monotone if it is either increasing or decreasing.*

Exercise 30. Prove that if f is strictly increasing, then it is one-to-one. What if f is only increasing?

Periodic functions.

Definition 70. *A function* $f: \mathbb{R} \mapsto \mathbb{R}$ *is said to be periodic, if and only if there is* $L > 0$ *such that*

$$f(x) = f(x + L) \tag{90}$$

holds for all $x \in \mathbb{R}$*. Such* L *is called a "period" of* f*.*

Exercise 31. Prove that a periodic function f always have infinitely many periods.

Remark 71. Often, among the infinitely many periods of f, there is a smallest one. In this case, when we talk about "the period of the function f", we are referring to this smallest period.

Exercise 32. What are the periods for $\sin x, \cos 3x, \tan 5x$?

Exercise 33. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be a periodic function and define the set A to be its periods, that is

$$A := \{L > 0 | \ f(x) = f(x + L) \text{ for all } x \in \mathbb{R}\}. \tag{91}$$

Let $T := \inf A$.

a) Prove that if $T > 0$, then $A = \{n T | \ n \in \mathbb{N}\}$, that is any other period is a multiple of T.

b) Consider the Dirichlet function

$$D(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}. \tag{92}$$

Prove that it is periodic but without smallest period.

A few more words about the definition of $A \Longrightarrow B$.

The definition for the logical statement $A \Longrightarrow B$ as false only when A is true and B is false is a bit puzzling. The following may help.[5]

Consider the claim

> If it rains, then I will bring an umbrella.

So here $A =$ it rains, and $B =$ I bring an umbrella. It is natural to write the above as $A \Longrightarrow B$. Now we consider, among the following four situations, which makes the claim false:

1. A true and B true: It rains, and I brought an umbrella;
2. A true and B false: It rains, and I didn't bring an umbrella;
3. A false and B true: It didn't rain, but I still brought an umbrella;
4. A false and B false: It didn't rain, and I didn't bring an umbrella.

I hope everyone agrees that the only situation that makes the original claim false is 2.

Bibliography

[Sib09] Thomas Q. Sibley. *Foundations of Mathematics*. John Wiley & Sons, 2009.

5. If anyone has better way of making sense of this definition, please either let me know or post on piazza or tell other students.