Math 314 Fall 2013 Midterm Solutions

Oct. 24, 2013 2pm - 3:20pm. Total 25 Pts

NAME: ID#:

- Please write clearly and show enough work.
- No electronic devices are allowed.

Question 1. (4 pts) *A sequence of functions $\{f_n(x)\}$ is said to converge uniformly on $E \subseteq \mathbb{R}$ to another function $f(x)$ if and only if:*

For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

a) **(1 pt)** *Write down the logical statement for the above (**all symbols, no words**);*

b) **(3 pts)** *Write down its working negation.*

Solution.

a) **(1 pt)** Answer is correct.

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n > N, \ \forall x \in E, \qquad |f_n(x) - f(x)| < \varepsilon. \tag{1}$$

b)

- **(1 pt)** : $\exists \leftrightarrow \forall$;
- **(1 pt)** : Keep $\varepsilon > 0$ etc.;
- **(1 pt)** : Final answer is correct.

$$\exists \varepsilon > 0, \ \forall N \in \mathbb{N}, \ \exists n > N, \ \exists x \in E, \qquad |f_n(x) - f(x)| \geqslant \varepsilon. \tag{2}$$

Question 2. (6 pts) *The sum of two sets of real numbers is defined as $A + B := \{a + b |\ a \in A, b \in B\}$. A function is* additive *if and only if $\forall a, b \in \mathbb{R}$, $f(a) + f(b) = f(a + b)$. Prove that*

$$f \text{ is additive} \Longleftrightarrow \forall A, B \subseteq \mathbb{R}, f(A + B) = f(A) + f(B). \tag{3}$$

Solution.

- **(1 pt)** Overall rigor of argument.

- **(1 pt)** Know need to prove $\Longrightarrow$ and $\Longleftarrow$;
- **(1 pt)** Know need to prove $\subseteq$ and $\supseteq$;
- $\Longrightarrow$. Assume f is additive. Take any $A, B \subseteq \mathbb{R}$.
    1. **(1 pt)** $f(A+B) \subseteq f(A) + f(B)$. Take any $y \in f(A+B)$, by definition there is $c \in A+B$ such that $y = f(c)$. Next by definition of $A+B$ there are $a \in A$, $b \in B$ such that $c = a + b$. As f is additive, $f(c) = f(a+b) = f(a) + f(b)$. By definition $f(a) \in f(A)$, $f(b) \in f(B)$. Again by definition $f(a) + f(b) \in f(A) + f(B)$. Therefore $y \in f(A) + f(B)$.
    2. **(1 pt)** $f(A) + f(B) \subseteq f(A+B)$. Take any $y \in f(A) + f(B)$. By definition $y = u + v$ where $u \in f(A), v \in f(B)$. By definition of image there are $a \in A, b \in B$ such that $u = f(a), v = f(b)$. Since f is additive, $f(a) + f(b) = f(a+b)$. By definition $a + b \in A + B$ therefore $y = f(a) + f(b) = f(a+b) \in f(A+B)$.
- $\Longleftarrow$. **(1 pt)** For any $a, b \in \mathbb{R}$, take $A = \{a\}$, $B = \{b\}$. Then $A + B = \{a+b\}$. We have

  $$\{f(a+b)\} = f(A+B) = f(A) + f(B) = \{f(a)\} + \{f(b)\} = \{f(a) + f(b)\} \tag{4}$$

  which means $f(a+b) = f(a) + f(b)$. So f is additive.

Question 3. (4 pts) *Prove* $\lim_{n \longrightarrow \infty} \left[\sqrt{n^4+n} - n^2\right] = 0$ ***by definition.***

Solution.

- **(2 pts)** Overall rigor and formalism;
- **(2 pts)** Correct argument;

We have

$$\sqrt{n^4+n} - n^2 = \frac{(n^4+n) - n^4}{\sqrt{n^4+n} + n^2} = \frac{n}{\sqrt{n^4+n} + n^2} < \frac{n}{n^2} = n^{-1}. \tag{5}$$

Now for any $\varepsilon > 0$, take $N > \varepsilon^{-1}$, then for any $n > N$,

$$\left|\sqrt{n^4+n} - n^2 - 0\right| = \frac{n}{\sqrt{n^4+n} + n^2} < n^{-1} < N^{-1} < \varepsilon. \tag{6}$$

Question 4. (4 pts)

a) **(2 pts)** *Prove that*

$$\sum_{n=1}^{\infty} \frac{26}{(n!)^2} \tag{7}$$

converges.

b) **(2 pts)** *Prove that*

$$\sum_{n=1}^{\infty} \frac{\sqrt{37}}{(n+2)\,(n+1)\,n} \tag{8}$$

converges.

Solution.

a) We apply the ratio test to $a_n = \frac{26}{(n!)^2}$. We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{26}{[(n+1)!]^2}}{\frac{26}{(n!)^2}} = \frac{(n!)^2}{[(n+1)!]^2} = \frac{1}{(n+1)^2}. \tag{9}$$

Since

$$0 < \frac{1}{(n+1)^2} < n^{-2} \tag{10}$$

we have

$$\limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n \to \infty} \frac{1}{(n+1)^2} = \lim_{n \to \infty} \frac{1}{(n+1)^2} = 0. \tag{11}$$

thanks to Squeeze theorem. By ratio test the series converges.

b) We have

$$\left| \frac{\sqrt{37}}{(n+2)\,(n+1)\,n} \right| < \frac{\sqrt{37}}{n^3} = \sqrt{37}\, n^{-3}. \tag{12}$$

Since $\sum_{n=1}^{\infty} \sqrt{37}\, n^{-3}$ converges, by the comparison theorem for series we conclude that $\sum_{n=1}^{\infty} \frac{\sqrt{37}}{(n+2)\,(n+1)\,n}$ converges.

Question 5. (5 pts) *Let* $x_n = (-1)^n + \frac{1}{n}$. *Find* $\sup_{n \in \mathbb{N}} \{x_n\}$, $\inf_{n \in \mathbb{N}} \{x_n\}$, $\limsup_{n \longrightarrow \infty} x_n$ *and* $\liminf_{n \longrightarrow \infty} x_n$

Solution.

- **(1 pt)** Overall rigor; **(2 pts)** Correct proofs for sup/inf; **(2 pts)** Correct proofs for limsup /liminf .

- $\sup_{n\in\mathbb{N}}\{x_n\}$. $\sup_{n\in\mathbb{N}}\{x_n\}=\frac{3}{2}$.
  - 3/2 is an upper bound. Take any $x_n\in\{x_n\}$. Two cases:
    1. $n\geqslant 2$. We have $(-1)^n\leqslant 1, \frac{1}{n}\leqslant\frac{1}{2}\Longrightarrow(-1)^n+\frac{1}{n}\leqslant\frac{3}{2}$;
    2. $n=1$. We have $(-1)^1+\frac{1}{1}=0\leqslant\frac{3}{2}$.
  - 3/2 is the least upper bound. Take any $b<\frac{3}{2}$. Then $x_2=(-1)^2+\frac{1}{2}=\frac{3}{2}>b$. Contradiction.
- $\inf_{n\in\mathbb{N}}\{x_n\}$. $\inf_{n\in\mathbb{N}}\{x_n\}=-1$.
  - -1 is a lower bound. Take any $x_n\in\{x_n\}$. Then

$$x_n=(-1)^n+\frac{1}{n}\geqslant(-1)^n\geqslant-1. \tag{13}$$

  - -1 is the greatest lower bound. Take any $b>-1$. There is $n\in\mathbb{N}$ such that $\frac{1}{n}<b+1$. Now

$$x_{2n+1}=(-1)^{2n+1}+\frac{1}{2n+1}=-1+\frac{1}{2n+1}<-1+\frac{1}{n}<b. \tag{14}$$

- $\text{limsup}_{n\longrightarrow\infty}x_n$. $\text{limsup}_{n\to\infty}x_n=1$.

  Take any $n\in\mathbb{N}$. Then we have $x_{2n}=(-1)^{2n}+\frac{1}{2n}>1$ therefore

$$\sup\{x_n,x_{n+1},\ldots\}>1. \tag{15}$$

  On the other hand for every $k\geqslant n$, we have

$$x_k\leqslant 1+\frac{1}{n}. \tag{16}$$

  Therefore

$$\sup\{x_n,x_{n+1},\ldots\}\leqslant 1+\frac{1}{n}. \tag{17}$$

  Applying Squeeze Theorem we have

$$\limsup_{n\to\infty}x_n=1. \tag{18}$$

- $\text{liminf}_{n\longrightarrow\infty}x_n$. $\text{liminf}_{n\to\infty}x_n=-1$.

  Take any $n\in\mathbb{N}$. Then we have $x_{2n+1}=(-1)^{2n+1}+\frac{1}{2n+1}=-1+\frac{1}{2n+1}$, therefore

$$\inf\{x_n,x_{n+1},\ldots\}\leqslant-1+\frac{1}{2n+1}. \tag{19}$$

On the other hand, for all $k \geqslant n$,

$$x_k = (-1)^k + \frac{1}{k} > (-1)^k \geqslant -1. \tag{20}$$

So

$$\inf \{x_n, x_{n+1}, \ldots\} \geqslant -1. \tag{21}$$

Applying Squeeze Theorem we have

$$\liminf_{n \to \infty} x_n = -1. \tag{22}$$

Question 6. (2 pts) *Let $f: \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Let $A := \{x \in \mathbb{R} | f(x) \geqslant 0\}$. Prove that A is closed.*

Solution. (1 pt) : Understand open/closed; **(1 pt)** Correct use of continuity;

We prove $A^c := \{x \in \mathbb{R} | f(x) < 0\}$ is open. Take any $x_0 \in A^c$. Then $f(x_0) < 0$. Denote $\varepsilon_0 = -f(x_0)$. Then by continuity of f at x_0 there is $\delta > 0$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon_0$.

Now as $f(x) - f(x_0) \leqslant |f(x) - f(x_0)|$ we have

$$f(x) \leqslant |f(x) - f(x_0)| + f(x_0) < \varepsilon_0 + f(x_0) = 0. \tag{23}$$

Therefore $\forall x \in (x_0 - \delta, x_0 + \delta)$, $f(x) < 0 \Longrightarrow x \in A^c$. So A^c is open and consequently A is closed.