# Contents



- This review may not cover all possible topics for the midterm exam. Please also review lecture notes and homework problems.

- To get the most out of these problems, *clearly write down* (instead of mumble or think) your *complete* answers (instead of a few lines of the main idea), in *full sentences* (instead of formulas connected by arrows). And then compare with the solutions when they are posted.

- If don't know where to start, write down all definitions involved.

- If have no idea what to do, try proof by contradiction. Start by writing down the assumption in logical statements.

- "Justify" means: if true, provide a proof; if false, give a counterexample.

## A. Propositional Logic: True or False

### 1. Concepts and theorems

- Mathematical statements: Either true or false.
- New statements can be created from old ones using
  - not: $\neg$ (Negation);

| $A$ | $\neg A$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

  - and: $\wedge$ (Conjunction);

| $A$ | $B$ | $A \wedge B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

  - or: $\vee$ (Disjunction);

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

  - implies: $\Longrightarrow$ (Conditional);

| $A$ | $B$ | $A \Longrightarrow B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

  - equivalent: $\Longleftrightarrow$ (Bi-conditional).

| $A$ | $B$ | $A \Longleftrightarrow B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

- To prove: Construct truth table.

**Exercise 1.** Prove that $A \Longrightarrow B$ is the same as $(\neg A) \vee B$.

**Note.** How to remember "If" and "Only if", "Sufficient" and "Necessary"...

"Only if" is opposite to "If". $A$ if $B$ means "If $B$ then $A$" that is $B \Longrightarrow A$, so "$A$ only if $B$" should be $A \Longrightarrow B$.

"Necessary" is opposite to "Sufficient". "$A$ is sufficient for $B$" is "$A \Longrightarrow B$", therefore "$A$ is necessary for $B$" is "$B \Longrightarrow A$".

**Note.** "If and only if". "$A$ if and only if $B$" means "$A$ if $B$" and "$A$ only if $B$", that is $B \Longrightarrow A$ and $A \Longrightarrow B$ and consequently $A \Longleftrightarrow B$.

### 2. Solutions to exercises

**Exercise 1**: Truth table:

| $A$ | $B$ | $A \Longrightarrow B$ | $\neg A$ | $(\neg A) \vee B$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

We see that $A \Longrightarrow B$ and $(\neg A) \vee B$ take the same truth values in all situations. In other words $(A \Longrightarrow B) \Longleftrightarrow ((\neg A) \vee B)$ is always true.

### 3. Problems

**Problem 1.** Let $A, B, C$ be logical statements. Prove that $[(A \Longrightarrow B)$ and $(B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C)$. Explain in English what this means.

**Problem 2.** Critique the following claim. Justify your answer.

If $(P \wedge Q) \Longrightarrow (R \vee S)$ and $Q \Longrightarrow R$, then $P \Longrightarrow S$.

## B. Sets

### 1. Concepts and theorems

- Set: A collection of objects.
- Important sets:
  - Empty set $\varnothing$: $\forall x, x \notin \varnothing$;
  - Natural numbers: $\mathbb{N} := \{1, 2, 3, \ldots\}$;
  - Integers: $\mathbb{Z}$;
  - Rational numbers: $\mathbb{Q}$;
  - Real numbers $\mathbb{R}$.
- Relations between an object and a set.
  - Object $x$ is a member of set $A$: $x \in A$;
  - Object $x$ is not a member of set $A$: $x \notin A$.
- Relations between sets.
  - Subset: $A \subseteq B$ $(B \supseteq A)$

    $$(x \in A) \Longrightarrow (x \in B). \tag{1}$$

    - Prove $A \subseteq B$: Take any $x \in A$, argueargueargue, $x \in B$.
    - Prove $A \nsubseteq B$: Find $x \in A$ but $x \notin B$.
  - Equal: $A = B$.

    $$(x \in A) \Longleftrightarrow (x \in B). \tag{2}$$

    - Prove $A = B$: Two steps.
      - Step 1. $A \subseteq B$;
      - Step 2. $B \subseteq A$.
    - Prove $A \neq B$: Find $x \in A$ but $x \notin B$, or find $x \in B$ but $x \notin A$.
  - Proper subset: $A \subset B$ $(B \supset A)$.

    $$(A \subseteq B) \wedge (A \neq B). \tag{3}$$

    - Prove $A \subset B$: Two steps.
      - Step 1. $A \subseteq B$;
      - Step 2. Find $x \in B$ but $x \notin A$.
- New sets from old.
  - Union:

    $$A \cup B := \{x |\, (x \in A) \vee (x \in B)\}. \tag{4}$$

  - Intersection:

    $$A \cap B := \{x |\, (x \in A) \wedge (x \in B)\}. \tag{5}$$

  - Subtraction:

    $$A - B := \{x |\, (x \in A) \wedge (x \notin B)\}. \tag{6}$$

  - Complement: Universal set $X$ – all sets under discussion are its subsets:

    $$A^c := \{x |\, x \notin A\}. \tag{7}$$

    This is a shorthand for a special case of subtraction.

**Exercise 2.** Prove

$$A \subseteq B \Longrightarrow A \cap C \subseteq B \cap C. \tag{8}$$

If $A \subset B$, can we conclude $A \cap C \subset B \cap C$? Justify.

- Intersection and union of arbitrary number os sets. Let $W$ be a collection of sets. Then

  $$\cap_{A \in W} A := \{x |\, \forall A \in W \quad x \in A\} \tag{9}$$

  $$\cup_{A \in W} A := \{x |\, \exists A \in W \quad x \in A\}. \tag{10}$$

**Note.** In particular, be aware of the difference between $\in$ and $\subseteq/\subset$. The former is about the relation between an element and a set (a collection of elements), while the latter is about the relation between two sets.

### 2. Solutions to exercises

**Exercise 2.** Take any $x \in A \cap C$. By definition of intersection $x \in A$ and $x \in C$. By definition of $A \subseteq B$ we have $x \in B$. Thus $x \in B$ and $x \in C$ and by definition of intersection $x \in B \cap C$.

If $A \subset B$ we cannot conclude $A \cap C \subset B \cap C$. For example $A = \{1\}, B = \{1, 2\}, C = \{1\}$.

### 3. Problems

**Problem 3.** Let $E_n := \{x \in \mathbb{R} |\, x > 1/n\}$. Calculate $\cup_{n \in \mathbb{N}} E_n$.

**Problem 4.** Let $A = \left\{x \in \mathbb{R} |\, |\sin x| \leqslant \frac{1}{2}\right\}; B = \{x \in \mathbb{R} |\, x^3 - x^2 + x - 1 < 0\}$.

- Represent $A, B, A \cup B, A \cap B$ using intervals.
- Which of these four sets is/are open? Which is/are closed? Justify your answers.

## C. Functions

### 1. Concepts and Theorems

- Function: A triplet consisting of two sets $A$, $B$ and a rule assigning to each element in $A$ one and only one element in $B$. Notation: $f: A \mapsto B$.
- Image and pre-image: $f: A \mapsto B$ a function.
  - $S \subseteq A$ has an image:
  $$f(S) := \{f(x) | x \in S\}. \tag{11}$$
  - $T \subseteq B$ has a pre-image:
  $$f^{-1}(T) := \{x | f(x) \in T\} \tag{12}$$

**Exercise 3.** Let $f: X \mapsto Y$ be function. Let $A, B \subseteq X$. Prove
$$f(A - B) \supseteq f(A) - f(B). \tag{13}$$
Give an example where $f(A - B) \supset f(A) - f(B)$.

- Composite function. $f$: $X \mapsto Y$, $g$: $Z \mapsto W$ functions. If $Y \subseteq Z$, can define a new function from $X$ to $W$, denoted $g \circ f$:
$$(g \circ f)(x) := g(f(x)). \tag{14}$$
- One-to-one, onto, bijection.
  - One-to-one: $\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$.
    - Prove one-to-one: Take any $a_1, a_2 \in A$. Assume $f(a_1) = f(a_2)$. ... $a_1 = a_2$.
  - Onto: $f(A) = B$.
    - Prove onto: Let $b \in B$ be arbitrary. We take $a = ....$, ..., $f(a) = b$.
  - Bijection: one-to-one and onto.
- Inverse funtion. $f: X \mapsto Y$ is a function.
  - Definition. $g$ is the inverse function of $f$ is and only if
    i. $g: Y \mapsto X$ is a function;
    ii. $\forall x \in X, g(f(x)) = x$;
    iii. $\forall y \in Y, f(g(y)) = y$.
  - $f$: $X \mapsto Y$ has inverse function if and only if it is a bijection.

**Exercise 4.** Suppose $f$: $A \mapsto B$ and $g$: $B \mapsto C$ are functions. Show that if both $f$ and $g$ are bijections, then so is $g \circ f$.

- Increasing, decreasing, monotone.
  - Increasing: $x_1 \geqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$.
  - Strictly increasing: $x_1 > x_2 \Longrightarrow f(x_1) > f(x_2)$.
  - Decreasing: $x_1 \geqslant x_2 \Longrightarrow f(x_1) \leqslant f(x_2)$.
  - Strictly decreasing: $x_1 > x_2 \Longrightarrow f(x_1) < f(x_2)$.
  - Monotone: Either increasing or decreasing.

### 2. Solutions to exercises

**Exercise 3.**

- Proof. Take any $y \in f(A) - f(B)$. By definition of set difference $y \in f(A)$ but $y \notin f(B)$. Now by definition of image there is $a \in A$ such that $y = f(a)$. If $a \in B$ then $y \in f(B)$ contradiction. Therefore $a \notin B$. So we have
$$a \in A, a \notin B \Longrightarrow a \in A - B. \tag{15}$$
- Example. Let $f$: $\mathbb{R} \mapsto \mathbb{R}$ be constant: $\forall x \in \mathbb{R}$, $f(x) = 0$. Let $A = \{1, 2\}$, $B = \{1\}$. Then
$$f(A - B) = f(\{2\}) = \{0\}, \text{ but} \tag{16}$$
$$f(A) - f(B) = \{0\} - \{0\} = \varnothing. \tag{17}$$

**Exercise 4.**

- $g \circ f$ is one-to-one. For any $x_1 \neq x_2$, since $f$ is one-to-one, $f(x_1) \neq f(x_2)$. Now because $g$ is one-to-one, $g(f(x_1)) \neq g(f(x_2))$.
- $g \circ f$ is onto. Take any $z \in C$. Since $g$ is onto, there is $y \in B$ such that $z = g(y)$. Now because $f$ is onto, there is $x \in A$ such that $y = f(x)$. Thus $z = g(f(x))$.

### 3. Problems.

**Problem 5.** Let $f: X \mapsto Y$ be a function. Prove that $f$ is one-to-one if and only if $f(A - B) = f(A) - f(B)$ for all subsets $A, B$ of $X$.

## D. Predicative Logic: Quantifiers

### 1. Concepts and theorems

- Universal quatifier: $\forall$.
  - Reads: $\forall x \in A\ P(x)$: "For any/every $x$ in $A$, the statement $P(x)$, when the variable takes this value $x$, is true."
  - Meaning: Can be understood as a "short hand".

    **Example.** $A := \{1, 2, 3\}$. $P(x)$ is "$x > 3$". Then $\forall x \in A\ P(x)$ means

    $$(1 > 3) \wedge (2 > 3) \wedge (3 > 3). \tag{18}$$

- Existential quantifier: $\exists$.
  - Reads: $\exists x \in A\ P(x)$: "There is $x$ in $A$ such that the statement $P(x)$, when the variable takes this value $x$, is true."
  - Meaning: Can be understood as a "short hand".

    **Example.** $A := \{1, 2, 3\}$. $P(x)$ is "$x > 3$". Then $\exists x \in A\ P(x)$ means

    $$(1 > 3) \vee (2 > 3) \vee (3 > 3). \tag{19}$$

- Working negation.
  - Try to "push" the "Not" through all quantifiers.
  - We can do this layer by layer.

    **Example.** To write the working negation of $\forall x \in A\ \exists y, z \in B\ P(x, y, z)$, we write

    $$\begin{aligned} & \neg[\forall x \in A\ \exists y, z \in B\ P(x, y, z)] \\ = {} & \exists x \in A\ \neg[\exists y, z \in B\ P(x, y, z)] \\ = {} & \exists x \in A \forall y, z \in B \quad \neg P(x, y, z). \end{aligned}$$

**Exercise 5.** Explain why the working negation of

$$\forall x > 0 \qquad f(x) > 0 \tag{20}$$

is

$$\exists x > 0 \qquad f(x) \leqslant 0 \tag{21}$$

instead of

$$\exists x \leqslant 0 \qquad f(x) \leqslant 0. \tag{22}$$

- To prove:
  - $\forall x \in A \quad P(x)$.

    Let $x \in A$ be arbitrary. [...some arguments here...], $P(x)$ is true.
  - $\exists x \in A \quad P(x)$. Two methods.
    1. Find such $x$ and show that $P(x)$ is true;
    2. Proof by contradiction. Assume

       $$\forall x \in A \quad \neg P(x) \tag{23}$$

       and reach contradiction.

**Note.** To obtain working negation correctly, the following steps should be followed:

1. Write all the quantifiers first.
2. When applying "not", $\forall$ becomes $\exists$, and $\exists$ becomes $\forall$.

### 2. Solutions to exercises

**Exercise 5.** $\forall x > 0\ x^2 > 0$ means

$$\wedge_{x > 0}(x^2 > 0) \tag{24}$$

that is

$$(x_1^2 > 0) \wedge (x_2^2 > 0) \wedge (x_3^2 > 0) \cdots \tag{25}$$

where $x_1$, $x_2$, $x_3$, ... lists all positive numbers (Note that more logic theory is needed to justify this).

Taking $\neg$:

$$\begin{aligned} & \neg[(x_1^2 > 0) \wedge (x_2^2 > 0) \wedge (x_3^2 > 0) \cdots] \\ = {} & \neg(x_1^2 > 0) \vee \neg(x_2^2 > 0) \vee \neg(x_3^2 > 0) \cdots \\ = {} & (x_1^2 \leqslant 0) \vee (x_2^2 \leqslant 0) \vee (x_3^2 \leqslant 0) \cdots \end{aligned}$$

which is (note that still the same $x_1, x_2, x_3, \ldots$)

$$\exists x > 0 \qquad x^2 \leqslant 0 \tag{26}$$

### 3. Problems

**Problem 6.** A function $f: [0, \infty) \mapsto \mathbb{R}$ is "Lipschitz" if and only if

$$\exists M > 0\ \forall x,\ y \geqslant 0 \qquad |f(x) - f(y)| \leqslant M\ |x - y|. \tag{27}$$

Write down the working negation of the above.

**Problem 7.** A function $f(x): \mathbb{R} \mapsto \mathbb{R}$ is increasing if $f(x_1) \geqslant f(x_2)$ whenever $x_1 \geqslant x_2$ Write down the logical statement for "$f(x)$ is not increasing".

## E. Sets of Real Numbers

### 1. Concepts and Theorems.

- Intervals: $[a, b], (a, b), [a, b), (a, b]$.
- Open sets:

$$\forall x \in A \ \exists (a, b) \ni x \qquad (a, b) \subseteq A \tag{28}$$

  - Prove $A$ is open: Take any $x \in A$. Find $a$, $b$ depending on $x$ and $A$ such that $x \in (a, b) \subseteq A$.
  - Prove $A$ is not open: Find $x \in A$, whenever $a < x < b$, there is $y \in (a, b)$, $y \notin A$.

  **Exercise 6.** Find a set $A$ that is not open but also not closed. Justify.

- Closed sets: $A$ is closed $\Longleftrightarrow$ $A^c$ is open.
  - To prove $A$ is closed: Prove $A^c$ is open.
  - To prove $A$ is not closed: Prove $A^c$ is not open.

**Theorem 1.** *Unions and intersections of open/closed sets.*

- sup and inf.
  - Intuition:
    * Sup: Best upper bound;
    * Inf: Best lower bound.
  - To prove $b = \sup A$. Two steps:
    * Step 1. Prove $b$ is an upper bound:

$$\forall a \in A, \qquad a \leqslant b \tag{29}$$

    * Step 2. Prove $b$ is the best, that is smallest, upper bound:

$$\forall b' < b \ \exists a \in A \qquad a > b'. \tag{30}$$

  - To prove $b = \inf A$. Two steps:
    * Step 1. Prove $b$ is a lower bound:

$$\forall a \in A, \qquad a \geqslant b \tag{31}$$

    * Step 2. Prove $b$ is the best, that is greatest, lower bound:

$$\forall b' > b \ \exists a \in A \qquad a < b'. \tag{32}$$

  - If $\sup A \in A$, it is also denoted $\max A$;
  - If $\inf A \in A$, it is also denoted $\min A$.

  **Exercise 7.** Let $A = \left\{\frac{n-2}{n} \middle| \ n \in \mathbb{N}\right\}$. Find $\sup A$. Justify your answer.

### 2. Solutions to Exercises.

**Exercise 6.** Take $A = [0, 1) := \{x \in \mathbb{R} | \ 0 \leqslant x < 1\}$.

- $A$ is not open. We take $0 \in A$. For any $a < 0 < b$, we have $a < \frac{a}{2} < 0 < b$. This gives

$$\frac{a}{2} \in (a, b) \text{ but } \frac{a}{2} \notin A. \tag{33}$$

- $A$ is not closed. We prove $A^c = (-\infty, 0) \cup [1, \infty)$ is not open. Take $1 \in A^c$. For any $a < 1 < b$, we have $b > 1 > \frac{1+a}{2} > a$ so

$$\frac{1+a}{2} \in (a, b) \text{ but } \frac{1+a}{2} \notin A^c. \tag{34}$$

**Exercise 7.** Guess $\sup A = 1$. Justify:

1. 1 is an upper bound of $A$. Take any $x \in A$. Then there is $n \in \mathbb{N}$ such that $x = \frac{n-2}{n} = 1 - \frac{2}{n} \leqslant 1$.
2. 1 is the best upper bound of $A$. Take any $b < 1$. There is $n \in \mathbb{N}$ such that $\frac{2}{n} < 1 - b$. Then

$$\frac{n-2}{n} = 1 - \frac{2}{n} > 1 - (1 - b) = b. \tag{35}$$

   So $b$ is not an upper bound of $A$.

### 3. Problems.

**Problem 8.** Let $A$ be a nonempty subset of $\mathbb{R}$. Let $B = 3\,A := \{3\,x : x \in A\}$. Derive the relations between $\sup B$, $\inf B$ and $\sup A$, $\inf A$. Justify your answers. Note that you may need to discuss different cases for $c$ and for $\sup A$.

## F. Limits of Sequences

### 1. Concepts and Theorems

- Definition

  $\lim_{n\longrightarrow\infty} x_n = L$ is defined as

  - $L \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|x_n - L| < \varepsilon$.
  - $L = \infty$. $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n > M$.
  - $L = -\infty$. $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n < M$.

  Observe the pattern.

- Calculating limits.

  - Tools:

    $\lim_{n\longrightarrow\infty} x_n = a$, $\lim_{n\longrightarrow\infty} y_n = b$ then

    a) $\lim_{n\longrightarrow\infty} (x_n \pm y_n) = a \pm b$;

    b) $\lim_{n\longrightarrow\infty} (x_n\, y_n) = a\, b$;

    c) If $b \neq 0$, $\lim_{n\longrightarrow\infty} (x_n / y_n) = a/b$.

- Proving existence of limits.

  - Definition.

    1. Guess the limit $L$.
    2. Proof: For any $\varepsilon > 0$, we take $N =$ [formula involving $\varepsilon$], then for all $n > N$, we have

    $$|x_n - L| \leqslant .... \leqslant \varepsilon. \tag{36}$$

  - Cauchy. If $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m, n > N$, $|x_n - x_m| < \varepsilon$, then $\lim_{n\rightarrow\infty} x_n$ exists.

    **Exercise 8.** Find a diverging sequence $x_n$ such that $\lim_{n\longrightarrow\infty} (x_{n+2} - x_n) = 0$.

  - Monotone.

    - Increasing. If

      1. $\forall n\ x_{n+1} \geqslant x_n$ (increasing);
      2. $\exists b\ \forall n\ x_n \leqslant b$ (upper bound);

      then $\lim_{n\rightarrow\infty} x_n$ exists.

    - Decreasing. If

      1. $\forall n\ x_{n+1} \leqslant x_n$ (decreasing);
      2. $\exists b\ \forall n\ x_n \geqslant b$ (lower bound);

      then $\lim_{n\rightarrow\infty} x_n$ exists.

  - Squeeze.

    1. $\exists N_0 \in \mathbb{N}\, \forall n > N_0 \quad w_n \leqslant x_n \leqslant y_n$;
    2. $\lim_{n\longrightarrow\infty} w_n = \lim_{n\longrightarrow\infty} y_n$.

    Then

    1. $\lim_{n\longrightarrow\infty} x_n$ exists;
    2. $\lim_{n\rightarrow\infty} x_n = \lim_{n\rightarrow\infty} w_n = \lim_{n\rightarrow\infty} y_n$.

- Comparing limits. If

  1. $\lim_{n\rightarrow\infty} x_n, \lim_{n\rightarrow\infty} y_n$ exist;
  2. $\exists N_0 \in \mathbb{N}\ \forall n > N_0 \qquad x_n \leqslant y_n$,

  then $\lim_{n\rightarrow\infty} x_n \leqslant \lim_{n\rightarrow\infty} y_n$.

### 2. Solutions to Exercises.

**Exercise 8.** Take $x_n = n^{1/2}$.

## G. Subsequence

### 1. Concepts and Theorems

- Subsequence.

$$\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, \ldots\} \tag{37}$$

is a subsequence of $\{x_n\} = \{x_1, x_2, \ldots\}$ if and only if

1. $\forall k \in \mathbb{N}$, $n_k \in \mathbb{N}$;

2. $n_1 < n_2 < n_3 < \ldots$.

**Exercise 9.** Let $\{x_n\}$ be a sequence. Prove: $\{x_n\}$ is bounded $\Longleftrightarrow$ Every subsequence of $\{x_n\}$ is bounded.

- limsup and liminf.
  - $\text{limsup}_{n \to \infty} x_n$ is
    * $\lim_{n \longrightarrow \infty} y_n$ where

$$y_n := \sup\{x_n, x_{n+1}, \ldots\}; \tag{38}$$

    * $\max A$ where $A$ is the set

$$\left\{a \in \mathbb{R} | \ \exists \{x_{n_k}\} \lim_{k \to \infty} x_{n_k = a}\right\} \tag{39}$$

  - $\text{liminf}_{n \longrightarrow \infty} x_n$ is
    * $\lim_{n \longrightarrow \infty} y_n$ where

$$y_n := \inf\{x_n, x_{n+1}, \ldots\}; \tag{40}$$

    * $\min A$ where $A$ is the set

$$\left\{a \in \mathbb{R} | \ \exists \{x_{n_k}\} \lim_{k \to \infty} x_{n_k = a}\right\} \tag{41}$$

  - How to calculate: Evaluating exactly $\sup_{k \geqslant n} x_k$ could be hard. There are two ways to overcome:
    * Use Squeeze theorem: Find $N_0 \in \mathbb{N}$ such that for all $n > N_0$,

$$w_n \leqslant \sup\{x_n, \ldots\} \leqslant z_n \tag{42}$$

    $\lim w_n = \lim z_n = L \Longrightarrow \text{limsup}_{n \to \infty} x_n = L$.

    **Exercise 10.** $x_n = (-1)^n + e^{-n^2}$.

    * Use limsup is the largest limit of convergent subsequences. First guess the limit $L$. Then show
      1. $\exists \{x_{n_k}\}$ converging to $L$.
      2. For every convergent subsequence $x_{n_k} \longrightarrow a$, $a \leqslant L$.

    **Exercise 11.** $x_n = (-1)^n + e^{-n^2}$.

- Some relations.
  - $\{x_n\}$ convergent $\Longrightarrow$ $\{x_n\}$ bounded; $\{x_n\}$ bounded $\Longrightarrow$ $\{x_n\}$ has a convergent subsequence;
  - $\{x_n\}$ convergent $\Longleftrightarrow$ All of its subsequences are convergent;
  - $\{x_n\}$ convergent $\Longleftrightarrow$ $\text{limsup}_{n \to \infty} x_n = \text{liminf}_{n \to \infty} x_n$.

### 2. Solutions to Exercises.

**Exercise 9.**

- $\Longrightarrow$. Since $\{x_n\}$ is bounded there is $M > 0$ such that $\forall n \in \mathbb{N}$ $|x_n| < M$. Since $n_k \in \mathbb{N}$, we have $\forall k \in \mathbb{N}$ $|x_{n_k}| < M$.

- $\Longleftarrow$. Assume $\{x_n\}$ is not bounded. Then for every $N \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $|x_{n_k}| \geqslant M$. The subsequence $\{x_{n_k}\}$ is then not bounded.

**Exercise 10.** We have

$$1 \leqslant \sup_{k \geqslant n} \left[(-1)^k + e^{-k^2}\right] \leqslant 1 + e^{-n^2}. \tag{43}$$

Taking limit $n \longrightarrow \infty$ we conclude

$$\text{limsup}\, x_n = 1. \tag{44}$$

**Exercise 11.**

1. Take $n_k = 2k$ then $x_{n_k} = 1 + e^{-4k^2} \longrightarrow 1$.

2. Comparison theorem:

$$x_{n_k} = (-1)^{n_k} + e^{-n_k^2} \leqslant 1 + e^{-k^2} \Longrightarrow a = \lim_{k \longrightarrow \infty} x_{n_k} \leqslant \lim_{k \longrightarrow \infty} \left(1 + e^{-k^2}\right) = 1. \tag{45}$$

### 3. Problems.

## H. Infinite Series

### 1. Concepts and Theorems.

- Definitions.
  - Infinite series: Formal summation
  $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \tag{46}$$
  - Convergence: Define partial sum
  $$s_n := \sum_{k=1}^{n} a_n := a_1 + \cdots + a_n. \tag{47}$$
  $\sum_{n=1}^{\infty} a_n$ convergens if and only if the sequence $\{s_n\}$ convergens. Call $\lim_{n\to\infty} s_n$ the "sum" of the infinite series.
- Convergence.
  - Definition: $\sum_{n=1}^{\infty} a_n = L$ if and only if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N$, $|\sum_{k=1}^{n} a_k - L| < \varepsilon$.
  - Convergence theorems: Adaptation of convergence theorems for sequences.
    * Cauchy criterion: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > m > N \quad \left|\sum_{m+1}^{n} a_k\right| < \varepsilon$.
    * Non-negative series: If $a_n \geqslant 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n \in \mathbb{R}$ if and only if $\{s_n\}$ is bounded from above.
    * Comparison: If $|a_n| \leqslant b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
  - $\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow \lim_{n\to\infty} a_n = 0$. But $\Longleftarrow$ is not true!
- Typical series.
  - Geometric. $\sum_{n=1}^{\infty} r^{n-1}$.
    * $|r| < 1 \Longrightarrow \sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$;
    * $r \geqslant 1 \Longrightarrow \sum_{n=1}^{\infty} r^{n-1} = \infty$;
    * $r \leqslant -1 \Longrightarrow \sum_{n=1}^{\infty} r^{n-1}$ does not converge.
  - Harmonic. $\sum_{n=1}^{\infty} n^{-a}$.
    * $a > 1 \Longrightarrow \sum_{n=1}^{\infty} n^{-a}$ converges;
    * $a \leqslant 1 \Longrightarrow \sum_{n=1}^{\infty} n^{-a} = \infty$.
- Convergence tests.
  - Ratio Test.
    * $\operatorname{limsup}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Longrightarrow$ converge;
    * $\operatorname{liminf}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Longrightarrow$ diverge;
    * Other situations $\Longrightarrow$ further study needed;
  - Root Test.
    * $\operatorname{limsup}_{n\to\infty} |a_n|^{1/n} < 1 \Longrightarrow$ converge;
    * $\operatorname{liminf}_{n\to\infty} |a_n|^{1/n} > 1 \Longrightarrow$ diverge;
    * Other situations $\Longrightarrow$ further study needed;

**Exercise 12.** Prove that $\sum_{n=1}^{\infty} n\, x^n$ converges when $|x| < 1$ and diverges when $|x| \geqslant 1$.

**Remark.** Keep in mind that if $\lim_{n\longrightarrow\infty} x_n$ exists, then $\operatorname{liminf}_{n\longrightarrow\infty} x_n = \operatorname{limsup}_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} x_n$.

**Remark.** Note that ratio/root tests are usually useless if the formulas for $a_n$ are not given.

### 2. Solutions to exercises.

**Exercise** 12. We apply the ratio test: Since $a_n = n\, x^n$ we have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} |x|$. We have
$$\lim_{n\longrightarrow\infty} \frac{n+1}{n} |x| = |x| \lim_{n\longrightarrow\infty} \frac{n+1}{n} = |x|. \tag{48}$$
Thus the ratio test gives:

$\sum_{n=1}^{\infty} n\, x^n$ converges when $|x| < 1$ and diverges when $|x| > 1$.

The case $|x| = 1$ has to be analyzed ad hoc. In this case we have $|a_n| = n$. Clearly $\lim_{n\longrightarrow\infty} |a_n| = 0$ doesn't hold. Therefore the series does not converge in this case.

### 3. Problems

**Problem 9.** Analyze the convegence/divergence of $\sum_{n=1}^{\infty} (x^n/n^2)$ for $x \in \mathbb{R}$.

# I. Limit of Functions

## 1. Concepts and Theorems

- $\lim_{x\longrightarrow a} f(x) = L$ is defined as
  - $a\in\mathbb{R}, L\in\mathbb{R}$. $\forall\varepsilon>0$, $\exists\delta>0$ such that $\forall 0<|x-a|<\delta$, $|f(x)-L|<\varepsilon$.
  - $a\in\mathbb{R}, L=\infty$. $\forall M\in\mathbb{R}$, $\exists\delta>0$ such that $\forall 0<|x-a|<\delta$, $f(x)>M$.
  - $a\in\mathbb{R}$, $L=-\infty$. $\forall M\in\mathbb{R}$, $\exists\delta>0$ such that $\forall 0<|x-a|<\delta$, $f(x)>M$.
  - $a=\infty$, $L\in\mathbb{R}$. $\forall\varepsilon>0$, $\exists M\in\mathbb{R}$ such that $\forall x>M$, $|f(x)-L|<\varepsilon$.
  - $a=-\infty, L=\infty$. $\forall M\in\mathbb{R}$, $\exists M'\in\mathbb{R}$ such that $\forall x<M'$, $f(x)>M$. Note that $M$ and $M'$ are not the same number.

  Observe the pattern.

  **Exercise 13.** Write definition for the following cases.

  1. $a=\infty, L=\infty$.
  2. $a=-\infty, L\in\mathbb{R}$.

- Left and right limits: For example $a, L\in\mathbb{R}$:
  - Right limit: $\lim_{x\longrightarrow a+} f(x) = L$ is defined as $\forall\varepsilon>0$, $\exists\delta>0$ such that $\forall 0<x-a<\delta$, $|f(x)-L|<\varepsilon$.
  - Left limit: $\lim_{x\longrightarrow a-} f(x) = L$ is defined as $\forall\varepsilon>0$, $\exists\delta>0$ such that $\forall -\delta<x-a<0$, $|f(x)-L|<\varepsilon$.

  **Exercise 14.** Write definition for $\lim_{x\to 0+} f(x) = -\infty$.

- Relation between function limit and sequence limit:

  > $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}$ with $x_n\neq a$ for all $n\in\mathbb{N}$, and $\lim_{n\to\infty} x_n = a$, there holds $\lim_{n\to\infty} f(x_n) = L$.

  **Exercise 15.** Prove that $\lim_{x\longrightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exist.

- Arithmetics: $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M$, then,

$$\lim_{x\to a} (f\pm g)(x) = L\pm M \tag{49}$$

$$\lim_{x\to a} (f g)(x) = L\, M, \tag{50}$$

$$\text{If } M\neq 0,\ \lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{L}{M}. \tag{51}$$

- Comparison: $h(x)\leqslant f(x)\leqslant g(x)$, $\lim_{x\to x_0} h(x) = L_1$, $\lim_{x\to x_0} f(x) = L_2$, $\lim_{x\to x_0} g(x) = L_3$, then $L_1\leqslant L_2\leqslant L_3$.
- Squeeze: $h(x)\leqslant f(x)\leqslant g(x)$, $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) = L$, then $\lim_{x\to x_0} f(x) = L$.

## 2. Solutions to exercises

**Exercise 13.**

1. $\forall M\in\mathbb{R}, \exists M'\in\mathbb{R}$ such that $\forall x>M'$, $f(x)>M$.
2. $\forall\varepsilon>0$, $\exists M\in\mathbb{R}$ such that $\forall x<M$, $|f(x)-L|<\varepsilon$.

**Exercise 14.**

$\forall M\in\mathbb{R}$, $\exists\delta>0$ such that for all $0<x<\delta$, $f(x)<M$.

**Exercise 15.** Take $x_n = \frac{1}{n\,\pi}$ and $y_n = \frac{1}{(2\,n+1/2)\,\pi}$. Then we have

$$\forall n, \qquad x_n\neq 0, y_n\neq 0; \tag{52}$$

$$\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n = 0. \tag{53}$$

But

$$\lim_{n\longrightarrow\infty} \sin\left(\frac{1}{x_n}\right) = \lim_{n\longrightarrow\infty} 0 = 0 \tag{54}$$

is different from

$$\lim_{n\longrightarrow\infty} \sin\left(\frac{1}{y_n}\right) = \lim_{n\longrightarrow\infty} 1 = 1. \tag{55}$$

## 3. Problems

**Problem 10.** Prove by definition that $\lim_{x\to a} f(x)$ exists and equals $L\in\mathbb{R}$ if and only if $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$ both exist and both equal $L$.

## J. Continuity/Continuous Functions

### 1. Continuity

- Definition: $\forall\varepsilon>0\exists\delta>0\forall|x-x_0|<\delta$, $|f(x)-f(x_0)|<\varepsilon$.
- Understanding.
  - Continuous at $x_0$:
    1. $\lim_{x\longrightarrow x_0}f(x)$ exists; **and**
    2. The limit equals $f(x_0)$.
  - Not continuous at $x_0$:
    1. $\lim_{x\longrightarrow x_0}f(x)$ does not exist, **or**
    2. it exists but is different from $f(x_0)$.
- Properties: $f,g$ continuous at $x_0$ then
  - $f\pm g$, $fg$ continuous at $x_0$;
  - If furthermore $g(x_0)\neq 0$, $f/g$ continuous at $x_0$.
- Composite functions.

  $f$ continuous at $x_0$, $g$ continuous at $y_0=f(x_0)$, then $g\circ f$ is continuous at $x_0$.
- Everyday functions:
  - Continuous at all $x_0\in\mathbb{R}$:
    * polynomials;
    * $\exp[x]$;
    * $\sin(x),\cos(x)$.
  - Rational functions: After cancelling common factors, continuous where $g\neq 0$, discontinuous where $g=0$.

### 2. Continuous functions

- Intermediate Value Theorem:

  Let $f(x)$ be continuous on the closed interval $[a,\,b]$. Then for every $s\in[f(a),\,f(b)]$ (or $[f(b),\,f(a)]$ if $f(b)\leqslant f(a)$), there is $\xi\in[a,b]$ such that $f(\xi)=s$.

  **Remark.** Note that $f(x)$ needs to be continuous on $[a,b]$, that is: For every $x_0\in[a,\,b]$, we have $\forall\varepsilon>0$, $\exists\delta>0$, $\forall x\in[a,\,b]$ $|x-x_0|<\delta$, $|f(x)-f(x_0)|<\varepsilon$. Or in other words:

  1. $\forall x_0\in(a,b)$, $\lim_{x\to x_0}f(x)=f(x_0)$;
  2. $\lim_{x\to a+}f(x)=f(a)$; $\lim_{x\to b-}f(x)=f(b)$.

  **Note.** If $f(x)$ is continuous on $(c,d)\supset[a,\,b]$, then $f(x)$ is continuous on $[a,b]$.

- Other consequences of $f$ continuous on $[a,\,b]$:
  - $f$ is bounded. There is $M>0$ such that $\forall x\in[a,b]$, $|f(x)|\leqslant M$.
  - $f$ reaches maximum and minimum. There are $x_{\max}$, $x_{\min}\in[a,\,b]$ such that $\forall x\in[a,b]$,

$$f(x_{\min})\leqslant f(x)\leqslant f(x_{\max}). \tag{56}$$

- Inverse function. $f\colon A\mapsto B$ satisfies
  1. continuous,
  2. onto,
  3. strictly increasing (or strictly decreasing)

  then the inverse $g$: $B\mapsto A$ exists and is continuous, onto, and strictly increasing (or strictly decreasing).

## K. Solutions

- **Problem 1.** We construct the truth table. Let AB denote $A \Longrightarrow B$, BC denote $B \Longrightarrow C$, AB BC denote $(A \Longrightarrow B) \wedge (B \Longrightarrow C)$, AC denote $A \Longrightarrow C$, $A...C$ denote $[(A \Longrightarrow B)$ and $(B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C)$.

| $A$ | $B$ | $C$ | AB | BC | AB BC | AC | $A...C$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

Therefore the statement is always true. It means if $A$ implies $B$ and $B$ implies $C$, then $A$ implies $C$.

- **Problem 2.** We can try to construct a truth table but we have four statements which means the table would have 16 rows. So instead we look at the claim

If $(P \wedge Q) \Longrightarrow (R \vee S)$ and $Q \Longrightarrow R$, then $P \Longrightarrow S$.

and decide that it looks wrong. Thus we need to assign truth values to $P$, $Q$, $R$, $S$ such that $(P \wedge Q) \Longrightarrow (R \vee S)$ and $Q \Longrightarrow R$ are true but $P \Longrightarrow S$ is false.

As $P \Longrightarrow S$ is false, we have to assigne $P = T$, $S = F$. Now to make $P \wedge Q = F$ we assign $Q = F$. Note that this impiles $(P \wedge Q) \Longrightarrow (R \vee S)$ and also $Q \Longrightarrow R$ are true.

- **Problem 3.** We prove $\cup_{n\in\mathbb{N}} E_n = \{x \in \mathbb{R} | x > 0\}$. Denote this set by $A$. We prove

1. $A \subseteq \cup_{n\in\mathbb{N}} E_n$. Take any $x \in A$. As $x > 0$, there is $n \in \mathbb{N}$ such that $x > \frac{1}{n}$ which means $x \in E_n \subseteq \cup_{n\in\mathbb{N}} E_n$.
2. $\cup_{n\in\mathbb{N}} E_n \subseteq A$. Take any $x \in \cup_{n\in\mathbb{N}} E_n$. By definition of union there is $n \in \mathbb{N}$ such that $x \in E_n$. This gives $x > \frac{1}{n} > 0$ therefore $x \in A$.

Summarizing, we have $\cup_{n\in\mathbb{N}} E_n = A$.

- **Problem 4.**

a) $A = \cup_{n\in\mathbb{Z}} \left[n\,\pi - \frac{\pi}{6}, n\,\pi + \frac{\pi}{6}\right]$; $B = \{x \in \mathbb{R}: (x-1)\,(x^2+1) < 0\} = (-\infty, 1)$. $A \cup B = (-\infty, 1) \cup \left(\cup_{n\in\mathbb{N}} \left[n\,\pi - \frac{\pi}{6}, n\,\pi + \frac{\pi}{6}\right]\right)$; $A \cap B = \cup_{n=0}^{\infty} \left[-n\,\pi - \frac{\pi}{6}, -n\,\pi + \frac{\pi}{6}\right]$.

b)

- $A$ is closed. Since $A^c = \cup_{n\in\mathbb{Z}} \left(n\,\pi + \frac{\pi}{6}, n\,\pi + \frac{5\,\pi}{6}\right)$ is open (because it is a union of open intervals).
- $B$ is open since it is an open interval.
- $C = A \cup B$ is neither open nor closed.
  - $*$ $C$ is not open. Take $x_0 = \frac{5\,\pi}{6} \in C$. Then for any $(a, b)$ such that $x_0 \in (a, b)$, there is $c > 0$ such that $\max\{1, a\} < c < x_0$. For this $c$ we have $c \notin A \cup B$. Consequently $(a, b) \nsubseteq A \cup B$.
  - $*$ $C$ is not closed. We have

$$(A \cup B)^c = \left[1, \frac{5\,\pi}{6}\right) \cup \left(\cup_{n=1}^{\infty} \left(n\,\pi + \frac{\pi}{6}, n\,\pi + \frac{5\,\pi}{6}\right)\right). \tag{57}$$

Now take $1 \in (A \cup B)^c$. For any $(a, b) \ni 1$, we have $a < \frac{1+a}{2} < 1$ and therefore $\frac{1+a}{2} \in (a, b)$ but $\frac{1+a}{2} \notin (A \cup B)^c$. Consequently $(a, b) \nsubseteq (A \cup B)^c$.

- $D = A \cap B$ is closed. Since $D^c = \left(\cup_{n=0}^{\infty} \left(-n\,\pi - \frac{5\,\pi}{6}, -n\,\pi - \frac{\pi}{6}\right)\right) \cup \left(\frac{\pi}{6}, \infty\right)$ is union of open intervals and is therefore open.

- **Problem 5.**
  - "If". Assume $\forall A,\ B \subseteq X,\ f(A\backslash B) = f(A)\backslash f(B)$. For any $x_1 \neq x_2$, take $A = \{x_1,\ x_2\}$, $B = \{x_2\}$. Then $f(A\backslash B) = \{f(x_1)\}$, $f(A) = \{f(x_1),\ f(x_2)\}$, $f(B) = \{f(x_2)\}$. As $f(A)\backslash f(B) = \{f(x_1)\}$, $f(x_1) \neq f(x_2)$.
  - "Only if". Assume $f$ is one-to-one. We prove $f(A\backslash B) \subseteq f(A)\backslash f(B)$ and $f(A)\backslash f(B) \subseteq f(A\backslash B)$.
    * $f(A\backslash B) \subseteq f(A)\backslash f(B)$. Take any $y \in f(A\backslash B)$. By definition there is $x \in A\backslash B$ such that $y = f(x)$. $x \in A\backslash B$ means $x \in A, x \notin B$.

      Because $x \in A$, $y = f(x) \in f(A)$; On the other hand, since $f$ is one-to-one and $x \notin B$, $y = f(x) \neq f(x')$ for any $x' \in B$ which means $y \notin f(B)$. Therefore $y \in f(A)\backslash f(B)$.
    * $f(A)\backslash f(B) \subseteq f(A\backslash B)$. Take any $y \in f(A)\backslash f(B)$. Then $y \in f(A)$, $y \notin f(B)$. As $y \in f(A)$ there is $x \in A$ such that $y = f(x)$. Since $y \notin f(B)$, $x \notin B$. Therefore $x \in A\backslash B$ and consequently $y = f(x) \in f(A\backslash B)$.

- **Problem 6.** The working negation is

$$\forall M > 0\ \exists x, y \geqslant 0 \quad |f(x) - f(y)| > M\,|x - y|. \tag{58}$$

- **Problem 7.** $f(x)$ is increasing if

$$\forall x_1, x_2\ x_1 \geqslant x_2 \quad f(x_1) \geqslant f(x_2). \tag{59}$$

  $f(x)$ is not increasing if

$$\exists x_1, x_2,\ x_1 \geqslant x_2, \qquad f(x_1) < f(x_2). \tag{60}$$

  Or simply write as

$$\exists x_1 \geqslant x_2 \qquad f(x_1) < f(x_2). \tag{61}$$

- **Problem 8.** We prove $\sup B = 3 \sup A$. We only need to show:
  1. $3 \sup A$ is an upper bound of $B$. For any $b \in B$, by definition there is $a \in A$ such that $b = 3\,a$. By definition of sup we have $\sup A \geqslant a \Longrightarrow 3 \sup B \geqslant 3\,a = b$.
  2. $3 \sup A$ is the best upper bound of $B$. Let $c < 3 \sup A$. Then $\frac{c}{3} < \sup A$. As $\sup A$ is the best upper bound for $A$, $\frac{c}{3}$ is not an upper bound for $A$. Therefore there is $a \in A$ such that $\frac{c}{3} < a$. This gives $c < 3\,a \in B$, that is $c$ is not an upper bound for $B$.

  $\inf B = 3 \inf A$ can be proved similarly.

- **Problem 9.** We have $a_n = \frac{x^n}{n^2}$ and therefore

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2}\,|x|. \tag{62}$$

  Since

$$\lim_{n \longrightarrow \infty} \frac{n^2}{(n+1)^2}\,|x| = |x|, \tag{63}$$

  the ratio test gives convergence for $|x| < 1$ and divergence for $|x| > 1$.

  For $|x| = 1$ we have

$$|a_n| = \frac{1}{n^2}. \tag{64}$$

  Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

- **Problem 10.**
  - If. Assume

$$\lim_{x \longrightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x) = L. \tag{65}$$

    Then for any $\varepsilon > 0$, there are $\delta_L, \delta_R > 0$ such that when $0 < x - a < \delta_R$ or $-\delta_L < x - a < 0$,

$$|f(x) - L| < \varepsilon. \tag{66}$$

    Now take $\delta = \min\{\delta_L, \delta_R\}$, we have

$$0 < |x - a| < \delta \implies \begin{array}{r} 0 < x - a < \delta_R \text{ or} \\ -\delta_L < x - a < 0 \end{array}$$

    Therefore for all $0 < |x - a| < \delta$,

$$|f(x) - L| < \varepsilon \tag{67}$$

    which means

$$\lim_{x \rightarrow a} f(x) = L. \tag{68}$$

- Only if.

  We prove first

$$\lim_{x\to a} f(x) = L \Longrightarrow \lim_{x\to a+} f(x) = L. \quad (69)$$

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow \quad (70)$$

$$|f(x) - L| < \varepsilon. \quad (71)$$

In particular

$$0 < x - a < \delta \Longrightarrow \quad (72)$$

$$|f(x) - L| < \varepsilon. \quad (73)$$

Next we prove

$$\lim_{x\to a} f(x) = L \Longrightarrow \lim_{x\to a-} f(x) = L. \quad (74)$$

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow \quad (75)$$

$$|f(x) - L| < \varepsilon. \quad (76)$$

In particular

$$-\delta < x - a < 0 \Longrightarrow \quad (77)$$

$$|f(x) - L| < \varepsilon. \quad (78)$$

Thus the proof ends.