Contents

- The Final is cumulative. Please also review material before midterm.
- This review may not cover all possible topics for the midterm exam. Please also review lecture notes and homework problems.
- To get the most out of these problems, *clearly write down* (instead of mumble or think) your *complete* answers (instead of a few lines of the main idea), in *full sentences* (instead of formulas connected by arrows). And then compare with the solutions when they are posted.
- If don't know where to start, write down all definitions involved.
- If have no idea what to do, try proof by contradiction. Start by writing down the assumption in logical statements.
- "Justify" means: if true, provide a proof; if false, give a counterexample.

L. Differentiation: Definitions

1. Concepts and theorems

- Definitions.
  - f is differentiable at $x_0 \in \mathbb{R}$:

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad (1)$$

    exists and is finite. (f is not differentiable at x_0 if the limit is ∞).
  - f is a differentiable function:

    f is differentiable at every x_0 in **its domain**.

    Example 1. $\sin x, \cos x, e^x, \ln x, 1/x$ are differentiable functions.

- Prove differentiability by definition.
  - Prove f is differentiable at x_0:
    1. Write $\frac{f(x) - f(x_0)}{x - x_0}$ or $\frac{f(x_0 + h) - f(x_0)}{h}$, simplify if possible;
    2. Prove that the limit $\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ or equivalently $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists and is finite.
  - Prove f is a differentiable function.

    Take any x_0 in the domain of f. Then prove f is differentiable at x_0.

    Exercise 1. Prove $f(x) = x^2$ is a differentiable function.

- Prove non-differentiability by definition.
  - Prove f is not differentiable at x_0:
    1. Write Write $\frac{f(x) - f(x_0)}{x - x_0}$ or $\frac{f(x_0 + h) - f(x_0)}{h}$, simplify if possible;
    2. Prove that the limit $\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ or equivalently $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ does not exist.
  - Prove f is not a differentiable function:
    1. Understand the behavior of f and make an educated guess of x_0.
    2. Prove f is not differentiable at x_0.

    Exercise 2. Let $f(x) = \begin{cases} x \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that $f(x)$ is not a differentiable function.

2. Solutions to exercises

EXERCISE 1. $f(x) = x^2$ is defined for all $x \in \mathbb{R}$ so its domain is $\mathbb{R}$. Take any $x_0 \in \mathbb{R}$, write

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = x + x_0. \qquad (2)$$

Taking limit $x \longrightarrow x_0$ we see

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 2\, x_0 \qquad (3)$$

is finite so f is differentiable at x_0. Therfore f is differentiable.

EXERCISE 2. By looking at the function (or by applying Chain rule) we realize that we should try $x_0 = 0$.

Write

$$\frac{f(x) - f(0)}{x - 0} = \cos\left(\frac{1}{x}\right). \qquad (4)$$

Taking $x_n = \frac{1}{2\, n\, \pi}$, $y_n = \frac{1}{(2\, n + 1)\, \pi}$ we have $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0$, $\forall n \in \mathbb{N}, x_n \neq 0, y_n \neq 0$, $\lim_{n \to \infty} \cos\left(\frac{1}{x_n}\right) = 1$, $\lim_{n \to \infty} \cos\left(\frac{1}{y_n}\right) = -1$ so $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ does not exist and therefore f is not differentiable at x_0. Consequently f is not a differentiable function.

3. Problems

Problem 1. Let $g(x)$ be differentiable at $x_0 = 0$ and $g(0) = 0$. Prove that $f(x) = \begin{cases} g(x) \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at $x_0 = 0$ if and only if $g'(0) = 0$.

Problem 2. Let $f(x) = |x + 1| + x$. Let $x_0 = -1$. Prove that $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2\, h} = 1$ but $f(x)$ is not differentiable at x_0.

M. Differentiation: Arithmetics

Here emphasize Leibniz rule and how to differentiate ratios.

1. Concepts and theorems

- Differentiability of sum, difference, product, ratio.

  Let f, g be differentiable at x_0. Then

  - $f \pm g$ is differentiable at x_0, with
  $$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0); \tag{5}$$
  - For $c \in \mathbb{R}$, $c\,f$ is differentiable at x_0, with
  $$(c\,f)'(x_0) = c\,f'(x_0). \tag{6}$$
  - **(Leibniz rule)** $f\,g$ is differentiable at x_0, with derivative
  $$f(x_0)\,g'(x_0) + f'(x_0)\,g(x_0). \tag{7}$$
  - If $g(x_0) \neq 0$ then f/g is differentiable at x_0 with derivative
  $$\frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{g(x_0)^2}. \tag{8}$$

Exercise 3. Apply Leibniz rule to $f = g\,(f/g)$ to derive (8).

Exercise 4. Let
$$f(x) = \frac{e^x \sin x}{\cos x}. \tag{9}$$
Calculate $f'(0)$.

2. Solutions to exercises

Exercise 3. Let $h(x) := \frac{f(x)}{g(x)}$. Then we have $f(x) = h(x)\,g(x)$ and Leibniz rule gives
$$\begin{aligned} f'(x_0) &= h'(x_0)\,g(x_0) + h(x_0)\,g'(x_0) \\ &= h'(x_0)\,g(x_0) + \frac{f(x_0)}{g(x_0)}\,g'(x_0) \end{aligned} \tag{10}$$
and (8) follows.

Exercise 4. We apply the ratio differentiation rule followed by Leibniz rule:
$$\begin{aligned} f'(x) &= \frac{(e^x \sin x)' \cos x - (e^x \sin x)\,(\cos x)'}{(\cos x)^2} \\ &= \frac{(e^x \sin x + e^x \cos x) \cos x}{(\cos x)^2} + \frac{e^x\,(\sin x)^2}{(\cos x)^2} \\ &= \frac{e^x\,(\sin x + \cos x)}{\cos x} + \frac{e^x\,(\sin x)^2}{(\cos x)^2}. \end{aligned}$$
Setting $x = 0$ we obtain $f'(0) = 1$.

3. Problems

N. Differentiation: Chain Rule

1. Concepts and theorems

- Chain rule: If
  1. f is differentiable at x_0;
  2. g is differentiable at $f(x_0)$,

  then $(g\circ f)(x):=g(f(x))$ is differentiable at x_0 and

$$(g\circ f)'(x_0)=g'(f(x_0))\,f'(x_0). \qquad (11)$$

Remark. Note that $g'(f(x_0))$ means first calculating the function g' and then evaluate it at the point $f(x_0)$.

Exercise 5. Prove that $f(x)=\exp\,[-1/x^3]$ is differentiable at every $x\neq 0$ and find $f'(x)$ there.

Exercise 6. Let $f(x)=\left(\frac{1+x^2}{1-x^2}\right)^3$. Calculate $f'(x)$ for $x\neq\pm 1$.

- Inverse function. If
  1. f is differentiable at x_0;
  2. g is the inverse function of f;
  3. $f'(x_0)\neq 0$,

  then $g(y)$ is differentiable at $y_0=f(x_0)$ with

$$g'(y_0)=\frac{1}{f'(x_0)}=\frac{1}{f'(g(y_0))}. \qquad (12)$$

Exercise 7. Let $f(x)=5\,x+\sin x$. Let $g(x)$ be its inverse function (for now we assume its existence). Calculate $g'(0)$.

Exercise 8. Let $f(x)=2\,x-\sin x$ defined on $\mathbb{R}$. Let $g(x)$ be its inverse function. Calculate $g'(0)$, $g'(\pi-1)$.

2. Solutions to exercises

Exercise 5. We know that e^x is differentiable at every $x\in\mathbb{R}$ and $-\frac{1}{x^3}$ is differentiable at every $x\neq 0$. Therefore the composite function exp $[-1/x^3]$ is differentiable at every $x\neq 0$.

Next calculate

$$\begin{aligned}(\exp\,[-1/x^3])' &= \exp'(-1/x^3)\,(-1/x^3)' \\ &= \exp(-1/x^3)\,[-(1/x^3)'] \\ &= \exp(-1/x^3)\,[3/x^4] \\ &= \frac{3\exp(-1/x^3)}{x^4}. \end{aligned} \qquad (13)$$

Exercise 6. We have

$$\begin{aligned} f'(x) &= 3\left(\frac{1+x^2}{1-x^2}\right)^2\left(\frac{1+x^2}{1-x^2}\right)' \\ &= 3\left(\frac{1+x^2}{1-x^2}\right)^2\left(\frac{4\,x}{(1-x^2)^2}\right) \\ &= \frac{12\,x\,(1+x^2)^2}{(1-x^2)^4}. \end{aligned} \qquad (14)$$

Exercise 7. We have

$$g'(0)=\frac{1}{f'(x_0)} \qquad (15)$$

where $x_0=g(0)$ or equivalently $f(x_0)=0$. Since $f(0)=0$ we see $x_0=0$.

$f'(x)=5+\cos x\Longrightarrow f'(0)=6$. So $g'(0)=\frac{1}{6}$.

Exercise 8. We have $f'(x)=2-\cos x\geqslant 1>0$ so g exists and is differentiable. We have

$$g'(y)=1/f'(x)=\frac{1}{2-\cos x} \qquad (16)$$

so all we need to do is to figure out x_1, x_2 such that $f(x_1)=0$ and $f(x_2)=\pi-1$. It's easily seen that $x_1=0$, $x_2=\pi/2$. Therefore

$$g'(0)=1, \qquad g'(\pi-1)=\frac{1}{2}. \qquad (17)$$

3. Problems

0. Differentiable Functions

1. Concepts and theorems

- f is differentiableat $x_0 \Longrightarrow f$ is continuous at x_0.
- Local maximizer/minimizer.
  - x_0 is local maximizer: $\exists \delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta),\ f(x) \leqslant f(x_0)$;
  - x_0 is local minimizer: $\exists \delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta),\ f(x) \geqslant f(x_0)$.
  - If
    1. x_0 is a local minimizer or maximizer for f;
    2. f is differentiable at x_0;

    then $f'(x_0) = 0$.

Exercise 9. Let $f(x) = x^2 \sin x$ Prove or disprove the following claim:

The local maximizers are $x = (2\,n + 1/2)\,\pi$ for $n \in \mathbb{Z}$.

- MVT: If
  1. f is continuous on $[a, b]$;
  2. f is differentiable on (a, b);

  Then $\exists \xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{18}$$

- Cauchy's MVT: If
  1. f, g are continuous on $[a, b]$,
  2. f, g are differentiable on (a, b),
  3. $g(a) \neq g(b)$,

  then $\exists \xi \in (a, b)$ such that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}. \tag{19}$$

- Monotonicity: f differentiable. Then
  - f increasing $\Longleftrightarrow f' \geqslant 0$;
  - f decreasing $\Longleftrightarrow f' \leqslant 0$;
  - $f' > 0 \Longrightarrow f$ strictly increasing;
  - $f' < 0 \Longrightarrow f$ strictly decreasing;
  - f is constant $\Longleftrightarrow f' = 0$.

  Note that f strictly increasing/decreasing on (a, b) does not imply $f' > 0 / < 0$ on (a, b)!

Exercise 10. Let $f(x) = 3\,x + x^3 + 2\sin x$. Prove that f is strictly increasing on $\mathbb{R}$.

2. Solutions to exercises

Exercise 9. The claim is false. Since $f(x)$ is differentiable, its local maximizers must satisfy $f'(x) = 0$:

$$0 = (x^2 \sin x)' = 2\,x \sin x + x^2 \cos x. \tag{20}$$

We check

$$f'(2\,n\,\pi + \pi/2) = (4\,n + 1)\,\pi \neq 0 \tag{21}$$

so $x = (2\ n\ +\ 1/2)\ \pi$ cannot be local maximizers.

Exercise 10. We calculate

$$f'(x) = 3 + 3\,x^2 + 2\cos x \geqslant 1 > 0 \tag{22}$$

so f is strictly increasing on $\mathbb{R}$.

3. Problems

Problem 3. Let f be continuous and differentiable on $\mathbb{R}$. If $\lim_{x \longrightarrow +\infty} f(x) = \lim_{x \longrightarrow -\infty} f(x)$, then there is $\xi \in \mathbb{R}$ such that $f'(\xi) = 0$.

Problem 4. Prove

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1 \tag{23}$$

for all $0 \leqslant x \leqslant \pi/2$. (Hint: Show $f(x) = \frac{\sin x}{x}$ is decreasing).

P. L'Hospital

1. Concepts and theorems

- Let $x_0 \in (a,b)$. Try to If
  1. $f(x), g(x)$ are differentiable on $(a,b) - \{x_0\}$;
  2. $\lim_{x\longrightarrow x_0} f(x) = \lim_{x\longrightarrow x_0} g(x) = 0$;
  3. $\lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)}$ exists;
  4. $g'(x) \neq 0$ for $x \in (a,b) - \{x_0\}$;

  Then

$$\lim_{x\longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)}. \tag{24}$$

Exercise 11. Calculate

$$\lim_{x\longrightarrow 0} \frac{\sin x - x}{x^3} \tag{25}$$

using L'Hospital's rule.

Exercise 12. Calculate

$$\lim_{x\longrightarrow 0} \frac{1-\cos^2 x}{\sqrt{1+x^2}-1} \tag{26}$$

using L'Hospital's rule.

2. Solutions to exercises

Exercise 11. Now that $\sin x - x$ and x^3 satisfies 1-4. Therefore the limit equals $\lim_{x\to 0} \frac{\cos x - 1}{3\,x^2}$ if the latter exists. Since $\cos x - 1$ and $3\,x^2$ still satisfies 1-4, the original limit would equal $\lim_{x\to 0} \frac{-\sin x}{6\,x}$ if this limit exists. As $-\sin x$ and $6\,x$ still satisfies 1-4, we can apply L'Hospital again to obtain

$$\lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}. \tag{27}$$

Therefore

$$\lim_{x\longrightarrow 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}. \tag{28}$$

Exercise 12. We first check that

$$\lim_{x\longrightarrow 0} (1 - \cos^2 x) = \lim_{x\longrightarrow 0} \left(\sqrt{1+x^2} - 1\right) = 0 \tag{29}$$

so we should apply L'Hospital's rule.

$$\begin{aligned}\lim_{x\longrightarrow 0} \frac{1-\cos^2 x}{\sqrt{1+x^2}-1} &= \lim_{x\longrightarrow 0} \frac{2\cos x \sin x}{x/\sqrt{1+x^2}} \\ &= \lim_{x\longrightarrow 0} \frac{2\cos x}{\sqrt{1+x^2}} \cdot \frac{\sin x}{x}.\end{aligned} \tag{30}$$

Notice that $\lim_{x\longrightarrow 0} \frac{2\cos x}{\sqrt{1+x^2}} = \frac{2}{1} = 2$. We only need to find $\lim_{x\longrightarrow 0} \frac{\sin x}{x}$. Applying L'Hospital's rule again:

$$\lim_{x\longrightarrow 0} \frac{\sin x}{x} = \lim_{x\longrightarrow 0} \frac{\cos x}{1} = 1. \tag{31}$$

So finally we conclude

$$\lim_{x\longrightarrow 0} \frac{1-\cos^2 x}{\sqrt{1+x^2}-1} = 2. \tag{32}$$

3. Problems

Q. Taylor Expansion

1. Concepts and theorems

- Higher order derivatives: Denote $f^{(0)}(x) = f(x)$.

  $\forall n \in \mathbb{N}$, $f(x)$ is n-th differentiable if and only if $f^{(n-1)}(x)$ exists and is differentiable at x_0. Denote

$$f^{(n)}(x_0) = \left(f^{(n-1)}\right)'(x_0). \tag{33}$$

Exercise 13. Let $n \in \mathbb{N}$. Let $f(x) = e^{2x}$. Calculate $f^{(n)}(0)$ for all $n \in \mathbb{N}$. Justify your answer.

- Let $f(x)$ be n-th differentiable.
  - Define its Taylor polynomial of degree n at x_0 as:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{34}$$

  - The difference:

$$R_n(x) = f(x) - P_n(x) \tag{35}$$

    is called the "remainder" at x_0.

  - $R_n(x)$ can be represented through several different formulas. The most popular one is the "Lagrange form" formula:

    If f is $(n+1)$-th differentiable, then

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \tag{36}$$

    where ξ satisfies $0 < |\xi - x_0| < |x - x_0|$.

Exercise 14. Calculate the Taylor expansion with Lagrange form of remainder to degree 2 at $x_0 = 0$ for $f(x) = \tan x$.

Exercise 15. Calculate Taylor expansion to degree 2 with Lagrange form of remainder at $x_0 = 1$ for

$$f(x) = \frac{1}{1+x^2}. \tag{37}$$

2. Solutions to exercises

Exercise 13. We prove by induction that $f^{(n)}(x) = 2^n e^{2x}$. Denote the claim by $P(n)$.

The base case $P(1)$: $f'(x) = 2\, e^{2x}$.

Assume $f^{(n)}(x) = 2^n e^{2x}$. By definition

$$f^{(n+1)}(x) = (2^n e^{2x})' = 2^{n+1} e^{2x}. \tag{38}$$

Thus $P(n) \Longrightarrow P(n+1)$.

Therefore $f^{(n)}(x) = 2^n e^{2x}$ and consequently $f^{(n)}(0) = 2^n$.

Exercise 14. We have

$$f(0) = \tan 0 = 0; \tag{39}$$

$$f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{(\cos x)^2}. \tag{40}$$

so

$$f'(0) = 1; \tag{41}$$

$$f''(x) = \left(\frac{1}{(\cos x)^2}\right)' = \frac{2 \sin x}{(\cos x)^3} \tag{42}$$

so $f''(0) = 0$;

$$f'''(x) = \frac{2}{(\cos x)^2} + 3\,\frac{2\,(\sin x)^2}{(\cos x)^4}. \tag{43}$$

Therefore the expansion is

$$\frac{\sin x}{\cos x} = x + \left[\frac{2\,(\cos \xi)^2 + 6\,(\sin \xi)^2}{(\cos \xi)^4}\right] \frac{x^3}{6}. \tag{44}$$

Exercise 15. We calculate:

$$f(1) = \frac{1}{2}; \tag{45}$$

$$f'(x) = -\frac{2\,x}{(1+x^2)^2} \Rightarrow f'(1) = -\frac{1}{2}. \tag{46}$$

$$f''(x) = \frac{6\,x^2 - 2}{(1+x^2)^3} \Longrightarrow f''(1) = \frac{1}{2}. \tag{47}$$

$$f'''(x) = \frac{24\,x\,(1-x^2)}{(1+x^2)^4}. \tag{48}$$

Therefore the expansion is

$$\frac{1}{2} - \frac{x-1}{2} + \frac{(x-1)^2}{4} + \frac{4\,\xi\,(1-\xi^2)}{(1+\xi^2)^4}(x-1)^3. \tag{49}$$

3. Problems

R. Definition of Riemann Integration

1. Concepts and Theorems.

- **(Partition)** Let $a,\, b \in \mathbb{R}$ with $a < b$. A partition of the interval $[a,\, b]$ is the set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_n = b. \tag{50}$$

- **(Upper/Lower Riemann sums)**

$$U(f, P) := \sum_{j=1}^{n} M_j(f)\,(x_j - x_{j-1}) \tag{51}$$

$$L(f, P) := \sum_{j=1}^{n} m_j(f)\,(x_j - x_{j-1}) \tag{52}$$

where

$$M_j(f) \;:=\; \sup_{x \in [x_{j-1}, x_j]} f(x) \tag{53}$$
$$m_j(f) \;:=\; \inf_{x \in [x_{j-1}, x_j]} f(x). \tag{54}$$

- **(Upper/lower Riemann integrals)**

$$U(f) := \inf_P \{U(f, P)\} \tag{55}$$

$$L(f) := \sup_P \{L(f, P)\}. \tag{56}$$

- **(Riemann integrability)**

  Integrable if and only if $U(f) = L(f)$. When integrable,

$$\int_a^b f(x)\,\mathrm{d}x = U(f) = L(f). \tag{57}$$

- Proving integrability by definition:

  Choose appropriate partitions P_n such that

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n). \tag{58}$$

Exercise 16. Prove by definition that

$$f(x) = \begin{cases} c & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{59}$$

is Riemann integrable on $[0, 1]$, no matter what c is.

2. Solutions to Exercises.

Exercise 16. Let $P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be such that $x_i = \frac{i}{n}$. Then we have

$$\inf_{x \in [x_i, x_{i+1}]} f(x) \geqslant -|c| \tag{60}$$

for $i = 0$ and $= 0$ for all other i's.

Similarly

$$\sup_{x \in [x_i, x_{i+1}]} f(x) \leqslant |c| \tag{61}$$

for $i = 0$ and $= 0$ for all other i's.

Therefore

$$-\frac{|c|}{n} \leqslant L(f, P_n) \leqslant U(f, P_n) \leqslant \frac{|c|}{n}. \tag{62}$$

By definition

$$L(f, P_n) \leqslant L(f) \leqslant U(f) \leqslant U(f, P_n). \tag{63}$$

Thus

$$-\frac{|c|}{n} \leqslant L(f) \leqslant U(f) \leqslant \frac{|c|}{n}. \tag{64}$$

Taking limit $n \to \infty$, by comparison theorem we have

$$0 \leqslant L(f) \leqslant U(f) \leqslant 0 \tag{65}$$

which gives $L(f) = U(f) = 0$ and integrability follows.

3. Problems.

Problem 5. Let $f(x)$ be integrable on $[a,\, b]$. Let $c \in \mathbb{R}$. Prove **by definition** that $c\,f(x)$ is integrable and $\int_a^b (c\,f)(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$. (Note that you need to discuss the sign of c)

Problem 6. Let $f(x)$ be integrable on $[a, b]$. Prove by definition of limit that

$$\lim_{x \to b-} \int_a^x f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x. \tag{66}$$

S. Criteria and properties

1. Concepts and Theorems

- Integrability: f is integrable on $[a, b]$ if and only if
  - For every $\varepsilon > 0$, there is a partition P such that $U(f, P) - L(f, P) < \varepsilon$; or
  - There is a sequence of partitions P_n such that $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$.
- f is integrable on $[a, b]$ if f is continuous on $[a, b]$.
  - Note that f needs to be continuous on the closed interval;
  - Note that the converse: f is integrable on $[a, b]$ only if f is continuous on $[a, b]$, is false.

  Exercise 17. Find a function f that is integrable on $[0, 1]$ but is not continuous on $[0, 1]$. Justify.
- Properties. Let $c \in \mathbb{R}$ and f, g be integrable on $[a, b]$. Then so are $|f|, c f, f \pm g, f g$.

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x;$$

$$\int_a^b (c f)(x)\,\mathrm{d}x = c \int_a^b f(x)\,\mathrm{d}x;$$

$$\int_a^b (f \pm g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x;$$

  Note that there is no relation between $\int_a^b f(x)\ g(x)\ \mathrm{d}x$ and $\left(\int_a^b f(x)\,\mathrm{d}x \right) \left(\int_a^b g(x)\,\mathrm{d}x \right)$.
- More properties. Let $a < c < b$. Then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$.
- Composite function. If f is integrable and g is continuous, then $g \circ f$ is integrable.

2. Solutions to Exercises

Exercise 17. Take $f = \begin{cases} 1 & x < 1/2 \\ 0 & x \geqslant 1/2 \end{cases}$. Obviously it is not continuous on $[0, 1]$. To justify its integrability, take $P_n = \left\{ 0, \frac{n-1}{2n}, \frac{1}{2}, 1 \right\}$. Then

$$L(f, P_n) = \frac{n-1}{2n},\ U(f, P_n) = \frac{1}{2}. \tag{67}$$

We have $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$ so f is integrable.

3. Problems

T. Fundamental Theorems of Calculus

1. Concepts and theorems

- FTC V1: If
  1. $f(x)$ is integrable on $[a,b]$;
  2. $F(x)$ is continuous on $[a,b]$;
  3. $\forall x\in(a,b), \quad F'(x)=f(x)$,

  then

$$\int_a^b f(x)\,\mathrm{d}x=F(b)-F(a). \tag{68}$$

- FTC V2:
  - Part I: If $f(x)$ is integrable on $[a,\, b]$, then

$$F(x):=\int_a^x f(t)\,\mathrm{d}t \tag{69}$$

    is continuous on $[a,b]$.
  - Part II: If
    1. $f(x)$ is integrable on $[a,b]$;
    2. $f(x)$ is continuous at $x_0\in(a,b)$,

    then $F(x)$ as defined above is differentiable at x_0 and

$$F'(x_0)=f(x_0). \tag{70}$$

Exercise 18. Let $F(x) := \int_{\sin x}^{x^2+2} e^t\,\mathrm{d}t$. Calculate $F'(x)$ and $F''(x)$.

- Integration by parts: If
  1. $u(x), v(x)$ are continuous on $[a,b]$;
  2. $u'(x), v'(x)$ are integrable on $[a,b]$;

  Then

$$\int_a^b u\,v'\,\mathrm{d}x=u(b)v(b)-u(a)v(a)-\int_a^b u'\,v\,\mathrm{d}x. \tag{71}$$

Exercise 19. Calculate

$$\int_0^1 x\,e^{3x}\,\mathrm{d}x. \tag{72}$$

- Change of variables: If
  1. $u(x)$ is continuous on $[a,b]$;
  2. $u(x)$ is differentiable on (a,b);
  3. $u'(x)$ is integrable on $[a,b]$;
  4. $f(y)$ is continuous on $I:=u([a,b])$;

  Then

$$\int_a^b f(u(t))\,u'(t)\,\mathrm{d}t=\int_{u(a)}^{u(b)} f(x)\,\mathrm{d}x. \tag{73}$$

2. Solutions to exercises

Exercise 19. Set $v=x, u'=e^{3x}\Longrightarrow u=e^{3x}/3$.

$$\begin{aligned}\int_0^1 x\,e^{3x}\,\mathrm{d}x &= \frac{x\,e^{3x}}{3}\Big|_0^1-\int_0^1\frac{e^{3x}}{3}\,\mathrm{d}x\\ &= \frac{e^3}{3}-\frac{e^3}{9}+\frac{e^0}{9}\\ &= \frac{1+2\,e^3}{9}.\end{aligned} \tag{74}$$

Exercise 18. Let $G(x) := \int_0^x e^t\,\mathrm{d}t$. Then we have $G'(x)=e^x$, and

$$F(x)=\int_0^{x^2+2} e^t\,\mathrm{d}t+\int_{\sin x}^0 e^t\,\mathrm{d}t=\int_0^{x^2+2} e^t\,\mathrm{d}t-\int_0^{\sin x} e^t\,\mathrm{d}t=G(x^2+2)-G(\sin x). \tag{75}$$

This gives

$$\begin{aligned}F'(x) &= G'(x^2+2)\,(x^2+2)'\\ &\quad -G'(\sin x)\,(\sin x)'\\ &= 2\,x\,e^{x^2+2}-e^{\sin x}\cos x.\end{aligned} \tag{76}$$

Taking derivative again we have

$$F''(x)=(4x^2+2)\,e^{x^2+2}+[\sin x-(\cos x)^2]\,e^{\sin x}. \tag{77}$$

3. Problems

Problem 7. Calculate the following integrals:

$$I_1=\int_e^{e^2}\frac{\mathrm{d}x}{x\,(\ln x)^4}; I_2=\int_0^4 e^{-\sqrt{x}}\mathrm{d}x; I_3=\int_1^e x^3\ln x\mathrm{d}x \tag{78}$$

Problem 8. Is the following calculation correct? Justify your answer.

$$\int_0^\pi \cos^2 x\,\mathrm{d}x=\int_0^0\frac{\mathrm{d}t}{(1+t^2)^2}=0 \tag{79}$$

where the change of variable is $t=\tan x$.

U. Improper Integrals

1. Concepts and Theorems.

- Definition.

  Let f: $(a, b) \mapsto \mathbb{R}$ is improperly integrable on (a,b) if and only if

  a) f is locally integrable: f is integrable on every $[c,d] \subset (a,b)$, and

  b) $\lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right)$ exists and is finite.

     Call this limit the improper Riemann integral of f over (a, b), denote it by

$$\int_a^b f(x)\,\mathrm{d}x. \quad (80)$$

Exercise 20. Prove that $x^{-1/3}$ is improperly integrable on $(0,1)$.

- Properties

  a) Integrable on $[a, b] \Longrightarrow$ improperly integrable on (a, b) and its improper integral equals its Riemann integral.

  b) If f is integrable on $[a, d]$ for every $d \in (a,b)$, then its improper integral

$$\int_a^b f(x)\mathrm{d}x = \lim_{d \longrightarrow b-} \int_a^d f(x)\,\mathrm{d}x \quad (81)$$

  If f is integrable on $[c, b]$ for every $c \in (a,b)$, then its improper integral

$$\int_a^b f(x)\mathrm{d}x = \lim_{c \longrightarrow a+} \int_c^b f(x)\,\mathrm{d}x \quad (82)$$

  c) If f is improperly integrable on (a,b), then the order of limit taking does not matter:

$$\begin{aligned} &\lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right) \\ = &\lim_{d \longrightarrow b-} \left(\lim_{c \longrightarrow a+} \int_c^d f(x)\,\mathrm{d}x \right) \\ = &\int_a^b f(x)\,\mathrm{d}x. \end{aligned} \quad (83)$$

Exercise 21. Prove that if f is integrable on $[a,b]$ then it is improperly integrable on (a,b).

2. Solutions to exercises.

Exercise 20. Take any $[c, d] \subset (0, 1)$. Since $x^{-1/3}$ is continuous on $[c, d]$, it is integrable on $[c, d]$. We calculate, through FTC Ver 1,

$$\int_c^d x^{-1/3}\,\mathrm{d}x = \frac{3}{2} x^{2/3}|_c^d = \frac{3}{2}\left[d^{2/3} - c^{2/3}\right]. \quad (84)$$

Now clearly

$$\lim_{c \to 0+} \left[\lim_{d \to 1-} \frac{3}{2}\left[d^{2/3} - c^{2/3}\right] \right] = \frac{3}{2}. \quad (85)$$

Exercise 21. First we know that if f is integrable on $[a, b]$, then it is integrable on every $[c, d] \subset (a, b)$. Now consider

$$F(x) := \int_a^x f(t)\,\mathrm{d}t. \quad (86)$$

By FTC Ver 2 we know that $F(x)$ is continuous on $[a,b]$. Thus

$$\begin{aligned} &\lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right) \\ = &\lim_{c \to a+} \left(\lim_{d \to b-} (F(d) - F(c)) \right) \\ = &\lim_{c \to a+} (F(b) - F(c)) \\ = &F(b) - F(a). \end{aligned} \quad (87)$$

Thus f is improperly integrable on (a, b). Finally by FTC Ver 1 we have

$$F(b) - F(a) = \int_a^b f(t)\,\mathrm{d}t. \quad (88)$$

3. Problems

Problem 9. Prove that, if $f(x)$ is improperly integrable on (a,b), then

$$\lim_{d \to b-} \left[\lim_{c \to a+} \int_c^d f(x)\,\mathrm{d}x \right] \quad (89)$$

exists and equals $\int_a^b f(x)\,\mathrm{d}x$.

Solutions to Problems

Problem 1.

- If. Since $g'(0) = 0$, by definition we have

$$\lim_{x\to 0} \frac{g(x)}{x} = 0 \Longrightarrow \lim_{x\to 0} \left|\frac{g(x)}{x}\right| = 0. \qquad (90)$$

Now we have

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| = \left|\frac{g(x)}{x}\sin\frac{1}{x}\right| \leqslant \left|\frac{g(x)}{x}\right|. \qquad (91)$$

By Squeeze Theorem we have

$$\lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0} - 0\right| = 0 \qquad (92)$$

which by definition gives $f'(0) = 0$.

- Only if.

We have

$$\lim_{x\to 0}\frac{g(x)}{x} = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = g'(0). \qquad (93)$$

Now we have

$$\frac{f(x)-f(0)}{x-0} = \frac{g(x)}{x}\sin\frac{1}{x}. \qquad (94)$$

Now take $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$. Then $x_n, y_n \neq 0$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Therefore

$$\lim_{n\to\infty}\frac{g(x_n)}{x_n} = \lim_{n\to\infty}\frac{g(y_n)}{y_n} = g'(0). \qquad (95)$$

Consequently

$$\lim_{n\to\infty}\frac{f(x_n)-f(0)}{x_n-0} = 0, \qquad (96)$$

$$\lim_{n\to\infty}\frac{f(y_n)-f(0)}{y_n-0} = g'(0). \qquad (97)$$

Since f is differentiable at 0, we must have $g'(0) = 0$.

Problem 2. For $h > 0$, we have

$$f(x_0+h) = h + (-1+h) = -1 + 2h; \qquad (98)$$

$$f(x_0-h) = h + (-1-h) = -1; \qquad (99)$$

For $h < 0$ we have

$$f(x_0+h) = (-h) + (-1+h) = -1; \qquad (100)$$

$$f(x_0-h) = (-h) + (-1-h) = -1 - 2h. \qquad (101)$$

Thus for all h we have

$$\frac{f(x_0+h)-f(x_0-h)}{2h} = 1 \qquad (102)$$

and the limit is obviously also 1.

Now for $h > 0$ we have

$$\frac{f(x_0+h)-f(x_0)}{h} = 2 \qquad (103)$$

while for $h < 0$ we have

$$\frac{f(x_0+h)-f(x_0)}{h} = 0. \qquad (104)$$

Therefore

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \qquad (105)$$

does not exist and f is not differentiable at x_0.

Problem 3. Denote

$$L := \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x). \qquad (106)$$

Three cases.

- $\sup_{x\in\mathbb{R}} f(x) = \inf_{x\in\mathbb{R}} f(x) = L$. Then $f(x) = L$ for all x and $f'(\xi) = 0$ for every $\xi \in \mathbb{R}$.

- $\sup_{x\in\mathbb{R}} f(x) > L$. Take $\{x_n\} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} f(x_n) = \sup_{x\in\mathbb{R}} f(x)$. Denote

$$\varepsilon_0 := \frac{\sup_{x\in\mathbb{R}} f(x) - L}{2}. \qquad (107)$$

Then

  - As $\lim_{x\to\infty} f(x) = L$, there is $R_1 > 0$ such that $|f(x) - L| < \varepsilon_0$ for all $x > R_1$;

- As $\lim_{x\to-\infty} f(x) = L$, there is $R_2 > 0$ such that $|f(x) - L| < \varepsilon_0$ for all $x < -R_2$;

- As $\lim_{n\to\infty} f(x) = \sup_{x\in\mathbb{R}} f(x)$, there is $N_0 \in \mathbb{N}$ such that $|f(x) - \sup_{x\in\mathbb{R}} f(x)| < \varepsilon_0$ for all $n > N_0$.

Summarizing, we have

$$\forall n > N_0, \qquad -R_2 \leqslant x_n \leqslant R_1. \tag{108}$$

By Bolzano-Weierstrass there is a subsequence x_{n_k} converging to some $\xi \in \mathbb{R}$. Since f is continuous,

$$f(\xi) = \lim_{k\to\infty} f(x_{n_k}) = \sup_{x\in\mathbb{R}} f(x). \tag{109}$$

Thus x_∞ is a maximizer of f and consequently $f'(\xi) = 0$.

Problem 4. Calculate

$$\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}. \tag{110}$$

To show that $f(x) = \frac{\sin x}{x}$ is decreasing, it's enough to show $g(x) = x\cos x - \sin x \leqslant 0$ for $0 \leqslant x \leqslant \pi/2$. Noticing that $g(0) = 0$, we calculate

$$g'(x) = \cos x - x\sin x - \cos x = -x\sin x < 0 \quad \text{for } 0 \leqslant x \leqslant \pi/2. \tag{111}$$

Therefore g is decreasing. Together with $g(0) = 0$ we have $g(x) = f'(x) \leqslant 0$. This implies $f(x)$ is decreasing. Therefore

$$\frac{2}{\pi} = \frac{\sin(\pi/2)}{\pi/2} \leqslant \frac{\sin x}{x} \leqslant \frac{\sin\delta}{\delta} \tag{112}$$

for all $x \in [\delta, \pi/2]$. Let $\delta \to 0$ we reach

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1 \tag{113}$$

for all $0 \leqslant x \leqslant \pi/2$.

Problem 5. We discuss the three cases $c > 0, c = 0, c < 0$. When $c = 0$ we have $c\,f = 0$ is constant which is integrable.

- $c > 0$. Let P be an arbitrary partition of $[a, b]$. We have

$$U(c\ f,\ P) = \sum_{j=1}^{n} \left[\sup_{[x_{j-1},x_j]} c\ f(x)\right] |x_j - x_{j-1}| = \sum_{j=1}^{n} c\left[\sup_{[x_{j-1},x_j]} f(x)\right] |x_j - x_{j-1}| = c\,U(f, P). \tag{114}$$

Taking infimum we have

$$U(c\,f) = \inf_P U(c\,f,\,P) = \inf_P\,[c\,U(f,\,P)] = c\inf_P U(f, P) = c\,U(f). \tag{115}$$

On the other hand,

$$L(c\ f,\ P) = \sum_{j=1}^{n} \left[\inf_{[x_{j-1},x_j]} c\ f(x)\right] |x_j - x_{j-1}| = c\,L(f, P) \tag{116}$$

which gives

$$L(c\,f) = c\,L(f). \tag{117}$$

As f is integrable, we have $U(f) = L(f) \Longrightarrow U(c\ f) = L(c\ f)$ so $c\ f$ is integrable with

$$\int_a^b (c\ f)(x)\ \mathrm{d}x = U(c\ f) = c\ U(f) = c\int_a^b f(x)\,\mathrm{d}x. \tag{118}$$

- $c < 0$. Let P be an arbitrary partition of $[a, b]$. We have

$$U(c\ f,\ P) = \sum_{j=1}^{n} \left[\sup_{[x_{j-1},x_j]} c\ f(x)\right] |x_j - x_{j-1}| = \sum_{j=1}^{n} c\left[\inf_{[x_{j-1},x_j]} f(x)\right] |x_j - x_{j-1}| = c\,L(f, P). \tag{119}$$

Taking infimum over all partitions we have

$$U(c\,f) = \inf_P U(c\,f,P) = \inf_P\,[c\,L(f,P)] = c\sup_P L(f,P) = c\,L(f). \qquad (120)$$

and similarly

$$L(c\,f) = c\,U(f). \qquad (121)$$

As f is integrable, we have $U(f) = L(f) \Longrightarrow U(c\;f) = L(c\;f)$ so $c\;f$ is integrable with

$$\int_a^b (c\;f)(x)\;\mathrm{d}x = U(c\;f) = c\;L(f) = c\int_a^b f(x)\,\mathrm{d}x. \qquad (122)$$

Problem 6. Since $f(x)$ is integrable on $[a,\;\;b]$, it is bounded on $[a,\;\;b]$. That is there is $M>0$ such that $\forall x\in[a,b], |f(x)|< M$. Now for any $\varepsilon > 0$, take $\delta := \frac{\varepsilon}{M+1}$. Then for any $0<b-x<\delta$, we have

$$\begin{aligned}\left|\int_a^x f(t)\,\mathrm{d}t - \int_a^b f(t)\,\mathrm{d}t\right| &= \left|\int_x^b f(t)\,\mathrm{d}t\right| \\ &\leqslant \int_x^b |f(t)|\,\mathrm{d}t \\ &< \int_x^b M\,\mathrm{d}t \\ &= M\,(b-x) \\ &< M\,\frac{\varepsilon}{M+1} \\ &< \varepsilon. \end{aligned} \qquad (123)$$

Problem 7.

- I_1. Change of variable: $y = u(x) = \ln x$. Then we have

$$\begin{aligned} I_1 = \int_e^{e^2} \frac{\mathrm{d}x}{x\,(\ln x)^4} &= \int_e^{e^2} \left(\frac{1}{u(x)^4}\right) u'(x)\,\mathrm{d}x \\ &= \int_{u(e)}^{u(e^2)} \frac{1}{y^4}\,\mathrm{d}y \\ &= \int_1^2 \frac{1}{y^4}\,\mathrm{d}y \\ &= -\frac{1}{3}\,y^{-3}|_1^2 \\ &= \frac{7}{24}. \end{aligned} \qquad (124)$$

- I_2. Change of variable: $y = u(x) = \sqrt{x}$. We have

$$\begin{aligned} I_2 = \int_0^4 e^{-\sqrt{x}}\,\mathrm{d}x &= \int_0^4 e^{-u(x)}\,u'(x)\,(2\,u(x))\,\mathrm{d}x \\ &= \int_{u(0)}^{u(4)} e^{-y}\,2\,y\,\mathrm{d}y \\ &= 2\int_0^2 y\,e^{-y}\,\mathrm{d}y \\ &= 2\int_0^2 y\,(-e^{-y})'\,\mathrm{d}y \\ &= 2\left[(-y\;\;e^{-y})|_0^2 + \int_0^2 e^{-y}\,\mathrm{d}y\right] \\ &= 2\,[-2\,e^{-2}+1-e^{-2}] \\ &= 2-6\,e^{-2}. \end{aligned} \qquad (125)$$

- I_3. We integrate by parts:

$$\begin{aligned} I_3 = \int_1^e x^3\ln x\,\mathrm{d}x &= \int_1^e \ln x\left(\frac{x^4}{4}\right)'\mathrm{d}x \\ &= \left[\ln x\left(\frac{x^4}{4}\right)\right]_{x=1}^{x=e} - \int_1^e \frac{x^4}{4}\,(\ln x)'\,\mathrm{d}x \\ &= \frac{e^4}{4} - \frac{1}{4}\int_1^e x^3\,\mathrm{d}x \\ &= \frac{3\,e^4+1}{16}. \end{aligned} \qquad (126)$$

Problem 8. No. Since $\cos^2 x > \frac{1}{2}$ when $x\in(0,\pi/4)$ we have

$$\int_0^{\pi} \cos^2 x\,\mathrm{d}x \geqslant \int_0^{\pi/4} \cos^2 x\,\mathrm{d}x > \int_0^{\pi/4} \frac{1}{2}\,\mathrm{d}x = \frac{\pi}{8} > 0 \qquad (127)$$

so the calculation is not correct.
The problem is $u(x) = \tan\;\; x$ is not differentiable over $(0,\pi)$.

Problem 9. We are given

$$\lim_{c\to a+}\left[\lim_{d\to b-}\int_c^d f(x)\,\mathrm{d}x\right] = L\in\mathbb{R}. \qquad (128)$$

Take any $x_0\in(a,b)$. Then we have

$$\int_c^d f(x)\mathrm{d}x = \int_c^{x_0} f(x)\mathrm{d}x + \int_{x_0}^d f(x)\mathrm{d}x. \qquad (129)$$

Thus the existence of

$$\lim_{d\to b-}\int_c^d f(x)\,\mathrm{d}x \tag{130}$$

implies the existence of

$$\lim_{d\to b-}\int_{x_0}^d f(x)\mathrm{d}x. \tag{131}$$

Denote it by $I(x_0)$. Then clearly

$$\lim_{c\to a+}\int_c^{x_0} f(x)\,\mathrm{d}x = L - I(x_0). \tag{132}$$

Now we have

$$\begin{aligned}\lim_{d\to b-}\left[\lim_{c\to a+}\int_c^d f(x)\,\mathrm{d}x\right] &= \lim_{d\to b-}\left[L - I(x_0) + \int_{x_0}^d f(x)\,\mathrm{d}x\right] \\ &= L - I(x_0) + \lim_{d\to b-}\int_{x_0}^d f(x)\,\mathrm{d}x \\ &= L - I(x_0) + I(x_0) \\ &= L. \end{aligned} \tag{133}$$