Math 314 Fall 2012 Midterm Review

Oct. 18, 2012

- Warning: This is **not** a complete list of materials covered by the midterm. You still need to read the notes, review homeworks, and work on practice problems.

1. Logic

- Logic: Understand the meaning of "and", "or", "not" as well as $\forall$ and $\exists$. In particular, know how to start a proof by contradiction.
    - When writing a logical statement, the quantifiers should come first: For example "$\exists M\ \forall x$ some statement" may sound similar to "$\exists M$ some statement $\forall x$", but the latter is not correct.
    - When applying "not", $\forall$ becomes $\exists$, and $\exists$ becomes $\forall$.

Example 1. A sequence of functions $f_n(x)\colon E \mapsto \mathbb{R}$ converges to another function $f(x)\colon E \mapsto \mathbb{R}$ *uniformly* if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$ and all $x \in E$, $|f(x) - f_n(x)| < \varepsilon$. In formal logic, the above reads

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ \forall x \in E, \qquad |f(x) - f_n(x)| < \varepsilon. \tag{1}$$

Therefore the statement "$f_n(x)$ does not converge uniformly to $f(x)$" should be written as

$$\exists \varepsilon_0 > 0\ \forall N \in \mathbb{N}\ \exists n > N\ \exists x \in E, \qquad |f(x) - f_n(x)| \geqslant \varepsilon_0. \tag{2}$$

2. Sets

- Sets: Understand set relations $\in, \ni, \subseteq, \supseteq, \subset, \supset, =$ and set operations $\cup, \cap, \backslash$ (the "complement" $\cdot^c$ is a special case of $\backslash$).
    - In particular, be aware of the difference between $\in$ and $\subseteq / \subset$. The former is about the relation between an element and a set (a collection of elements), while the latter is about the relation between two sets. Therefore if A is a set, "take $y \in A$" makes sense, but "take $y \subseteq A$" does not – unless your "y" is a set, but such use is discouraged because traditionally a, b, c, x, y, z are elements while A, B, C, X, Y, Z are sets.
- Sets in $\mathbb{R}$:
    - Intervals: $[a, b], (a, b), [a, b), (a, b]$.
    - Open sets: A set $A \subseteq \mathbb{R}$ is open if for every $x \in A$, there is an open interval $(a, b) \subseteq A$ such that $x \in (a, b)$. In logical statement:

$$\forall x \in A\ \exists (a, b) \text{ such that } x \in (a, b) \text{ and } (a, b) \subseteq A. \tag{3}$$

        - $A \subseteq \mathbb{R}$ not open:

$$\exists x_0 \in A\ \forall (a, b)\ [\text{Not } (x \in (a, b) \text{ and } (a, b) \subseteq A)] \tag{4}$$

        which for practical purposes becomes:

$$\exists x_0 \in A\ \forall (a, b) \ni x\ \ (a, b) \not\subseteq A. \tag{5}$$

    - Closed sets: A set $A \subseteq \mathbb{R}$ is closed if A^c is open.
    - Important: Not open $\not\Longleftrightarrow$ closed; Not closed $\not\Longleftrightarrow$ open. For example $(0, 1] = \{x \in \mathbb{R}\colon 0 < x \leqslant 1\}$ is neither open nor closed. (Think: How to prove this?)
    - Unions and intersections of open/closed sets.
    - sup and inf.
        - $A \subseteq \mathbb{R}$ nonempty. Then $\sup A$ and $\inf A$ exists (may be $\infty, -\infty$). While $\max A, \min A$ may not exist.

- To show $b = \sup A$, show
  1. b is an upper bound of A: $\forall a \in A, b \geqslant a$.
  2. If $\tilde{b} \in \mathbb{R}$ is an upper bound of A, then $b \leqslant \tilde{b}$.

  To show $b = \inf A$, show
  1. b is an lower bound of A: $\forall a \in A, b \leqslant a$.
  2. If $\tilde{b} \in \mathbb{R}$ is an lower bound of A, then $b \geqslant \tilde{b}$.
- To show $b = \max A$ ($\min A$). show 1. b is an upper bound of A; 2. $b \in A$.
- If $\sup A \in A$, then $\max A$ exists and equals $\sup A$; On the other hand, if $\max A$ exists, then $\sup A = \max A$. Similar statement for $\inf A$ and $\min A$.

3. FUNCTIONS

- $f: X \mapsto Y$: relation, domain, range.

  Example 2. (Sequences and subsequences) A real sequence is a function f: $\mathbb{N} \mapsto \mathbb{R}$. A subsequence is a composite function $g \circ f: \mathbb{N} \mapsto \mathbb{R}$ with $f: \mathbb{N} \mapsto \mathbb{N}$, $g: \mathbb{N} \mapsto \mathbb{R}$.

- Image and pre-image:
  - $f: X \mapsto Y$. $A \subseteq X$. Image $f(A) := \{f(a): a \in A\} = \{b \in Y: \exists a \in A \ b = f(a)\}$.
  - $f: X \mapsto Y$. $B \subseteq Y$. Pre-image $f(B) := \{a \in X: f(a) \in B\}$.
- One-to-one, onto, bijection, inverse function.
  - $f: X \mapsto Y$ is one-to-one: $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.
  - $f: X \mapsto Y$ is onto: $f(X) = Y$.
  - $f: X \mapsto Y$ is a bijection: one-to-one and onto.
  - $g: Y \mapsto X$ is an inverse function of f if both of the following hold:
    1. $\forall x \in X, \ g(f(x)) = x$.
    2. $\forall y \in Y, \ f(g(y)) = y$.
- Increasing, decreasing, monotone.
  - $f(x)$ is increasing: $x_1 \geqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$. Strictly increasing: $x_1 > x_2 \Longrightarrow f(x_1) > f(x_2)$.
  - $f(x)$ is decreasing: $x_1 \geqslant x_2 \Longrightarrow f(x_1) \leqslant f(x_2)$. Strictly decreasing: $x_1 > x_2 \Longrightarrow f(x_1) < f(x_2)$.
  - Monotone: Either increasing or decreasing.
- Set operations under functions: See homework.

4. LIMIT

- A unified approach. $f(x) \longrightarrow L \in \mathbb{R} \cup \{\infty, -\infty\}$ as $x \longrightarrow a \in \mathbb{R} \cup \{\infty, -\infty\}$.
  - $\longrightarrow a \in \mathbb{R}$. Distance to a can be arbitrarily small;
  - $\longrightarrow \infty$. Eventually larger than any number;
  - $\longrightarrow -\infty$. Eventually smaller than any number.
- Definitions of various convergence can be "assembled": $\lim_{x \longrightarrow a} f(x) = L$ where a, L can both be extended real numbers. In the following red corresponds to red.

  Example 3. Limit of sequence $\{x_n\}$. Recall that a sequence is a function f: $\mathbb{N} \mapsto \mathbb{R}$. The only possibility is $n \longrightarrow \infty$. Therefore $\lim_{n \longrightarrow \infty} x_n = L$ is defined as

  - $L \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|x_n - L| < \varepsilon$. Here $\forall \varepsilon > 0, \ldots, |x_n - L| < \varepsilon$ quantifies "f arbitrarily close to L" while $\exists N \in \mathbb{N}$ such that $\forall n > N$ quantifies "when n is arbitrarily large" (when n is arbitrarily "close" to ∞).

  - $L = \infty$. $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n > M$.
  - $L = -\infty$. $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n < M$.

Example 4. Limit of real function. $\lim_{x \longrightarrow a} f(x) = L$ is defined as

  - $a \in \mathbb{R}, L \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
  - $a \in \mathbb{R}, L = \infty$. $\forall M \in \mathbb{R}$, $\exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $f(x) > M$.
  - $a \in \mathbb{R}, L = -\infty$. $\forall M \in \mathbb{R}$, $\exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $f(x) > M$.
  - $a = \infty$, $L \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists M \in \mathbb{R}$ such that $\forall x > M$, $|f(x) - L| < \varepsilon$.
  - $a = -\infty, L = \infty$. $\forall M \in \mathbb{R}$, $\exists M' \in \mathbb{R}$ such that $\forall x < M'$, $f(x) > M$. Note that M and M' are not the same number.

The definition for other cases can be designed similarly.

- To prove the existence of limit $\lim_{x \longrightarrow a} f(x)$:
  - Use definition (when the limit is given or you can guess it).
  - Show it's Cauchy: Notice the similarity between red statements.
    - If f is a sequence $\{x_n\}$: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall m, n > N$, $|x_m - x_n| < \varepsilon$;
    - If f is a real function $f(x)$:
      - $a \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall 0 < |x_1 - a|, |x_2 - a| < \delta$, $|f(x_1) - f(x_2)| < \varepsilon$.
      - $a = \infty$. $\forall \varepsilon > 0$, $\exists M > 0$, $\forall x_1, x_2 > M$, $|f(x_1) - f(x_2)| < \varepsilon$.
    - The idea of "Cauchy" is only meaningful when the limit is finite. (Try to define "Cauchy" for $x_n \longrightarrow \infty$ to see why)
  - Show it's monotone (increasing or decreasing). The following is true for both sequences and functions. (Think: How to prove?)
    - Increasing + bounded above $\Longrightarrow$ $\lim = L \in \mathbb{R}$; Decreasing + bounded below $\Longrightarrow$ $\lim = L \in \mathbb{R}$.
    - Increasing + not bounded above $\Longrightarrow$ $\lim = \infty$; Decreasing + not bounded below $\Longrightarrow$ $\lim = -\infty$.
- From functions to sequences and back:
  - $\lim_{x \longrightarrow a} f(x) = L \Longleftrightarrow$ for every sequence $x_n \longrightarrow a$, $f(x_n) \longrightarrow L$ as $n \longrightarrow \infty$.
  - This holds even if a, L are extended real numbers.
- Properties:
  - At most one limit.
  - Arithmetic $+, -, \times, \div$. Note that the limit of the denominator should not be zero.
  - Comparison and Squeeze.
  - Composite function and inverse function.
- Subsequences:
  - A subsequence of $\{x_n\}$ is a sequence $\{y_k\}$ such that $y_k = x_{n_k}$. Here n_k is a sequence of natural numbers, that is $n_k: \mathbb{N} \mapsto \mathbb{N}$.
  - Bolzano-Weierstrass: Any bounded sequence has at least one converging subsequence.
- $\liminf$ and $\limsup$.
  - Limits of $\inf$ and $\sup$.

- Definition:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geqslant n} x_k = \lim_{n\longrightarrow\infty} \sup\{x_n, x_{n+1}, \ldots\}. \tag{6}$$

$$\liminf_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} \inf_{k\geqslant n} x_k = \lim_{n\longrightarrow\infty} \inf\{x_n, x_{n+1}, \ldots\}. \tag{7}$$

  In other words, we define a new sequence by

$$\begin{aligned} y_1 &= \sup\{x_1, x_2, x_3, \ldots\} & (8)\\ y_2 &= \sup\{x_2, x_3, x_4, \ldots\} & (9)\\ &\vdots \\ y_n &= \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} & (10)\\ &\vdots \end{aligned}$$

  and then define

$$\limsup_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n. \tag{11}$$

- Understanding:

$$\limsup_{n\longrightarrow\infty} x_n = \max\{a\in\mathbb{R}: \exists\text{subsequence } x_{n_k}\longrightarrow a\}; \tag{12}$$

$$\liminf_{n\longrightarrow\infty} x_n = \min\{a\in\mathbb{R}: \exists\text{subsequence } x_{n_k}\longrightarrow a\}. \tag{13}$$

- How to calculate: Evaluating exactly $\sup_{k\geqslant n} x_k$ could be hard. There are two ways to overcome:
  - Use Squeeze theorem:
    Instead of evaluating $\sup_{k\geqslant n} x_n$ exactly, find an upper bound and an lower bound

$$w_n \leqslant \sup\{x_n, \ldots\} \leqslant z_n \tag{14}$$

    and try to show $\lim w_n = \lim z_n$. Note that the requirement for z_n is $z_n \geqslant x_k$ for all $k \geqslant n$ while the requirement for w_n is $w_n \leqslant x_k$ for some $k \geqslant n$.

    Example 5. Consider $x_n = (-1)^n + e^{-n^2}$. We have

$$1 \leqslant \sup_{k\geqslant n}\left[(-1)^k + e^{-k^2}\right] \leqslant 1 + e^{-n^2}. \tag{15}$$

    Taking limit $n \longrightarrow \infty$ we conclude

$$\limsup x_n = 1. \tag{16}$$

  - Use limsup is the largest limit of convergent subsequences. To take this approach we need to first guess $\limsup_{n\longrightarrow\infty} x_n = 1$. Then show
    1. There is a subsequence converging to 1. Take $n_k = 2\,k$ then $x_{n_k} = 1 + e^{-4k^2} \longrightarrow 1$.
    2. For every convergent subsequence $x_{n_k} \longrightarrow a$, $a \leqslant 1$. We do this through comparison theorem:

$$x_{n_k} = (-1)^{n_k} + e^{-n_k^2} \leqslant 1 + e^{-k^2} \Longrightarrow a = \lim_{k\longrightarrow\infty} x_{n_k} \leqslant \lim_{k\longrightarrow\infty}\left(1 + e^{-k^2}\right) = 1. \tag{17}$$

5. CONTINUOUS FUNCTIONS

- Continuous at x_0: $f(x_0) = \lim_{x\to x_0} f(x)$.
- Continuous: A function is "continuous" (without "at x_0") if it is continuous at every point in its domain.

  Example 6. $e^x: \mathbb{R} \mapsto \mathbb{R}$ is continuous; $\ln x: (0, \infty) \mapsto \mathbb{R}$ is continuous.

- Continuity under $+, -, \times, \div$, composition and inverse.