Math 314 Fall 2012 Midterm Practice Problems

Oct. 15, 2012

- To best prepare for midterm, also review homework problems.
- To get most out of these problems, *clearly write down* (instead of mumble or think) your *complete* answers (instead of a few lines of the main idea), in *full sentences* (instead of formulas connected by arrows). And then compare with the solutions when they are posted.
- If don't know where to start, write down all definitions involved.
- If have no idea what to do, try proof by contradiction. Start by writing down the assumption in logical statements.
- "Justify" means: if true, provide a proof; if false, give a counterexample.

Practice Problems

Problem 1. $f(x): E \mapsto \mathbb{R}$ is Hölder continuous if there are $a > 0$ and $M \in \mathbb{R}$ such that for every $x, y \in E$, $|f(x) - f(y)| \leqslant M\,|x - y|^a$. Write down the logical statement for "$f(x)$ is not Hölder continuous".

Problem 2. Recall that $f(x)$ is increasing if $f(x_1) \geqslant f(x_2)$ whenever $x_1 \geqslant x_2$ Write down the logical statement for "$f(x)$ is not increasing".

Problem 3. Let A, B, C be logical statements. Prove that $[(A \Longrightarrow B)$ and $(B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C)$. Explain in English what this means.

Problem 4. Let $f: X \mapsto Y$ be a function. Prove that f is one-to-one if and only if $f(A \backslash B) = f(A) \backslash f(B)$ for all subsets A, B of X.

Problem 5. Suppose $f: A \mapsto B$ and $g: B \mapsto C$ are functions. Show that if both f and g are bijections, then so is $g \circ f$.

Problem 6. Let

$$A = \left\{x \in \mathbb{R}: |\sin x| \leqslant \frac{1}{2}\right\}; \qquad B = \{x \in \mathbb{R}: x^3 - x^2 + x - 1 < 0\}. \tag{1}$$

a) Represent $A, B, A \cup B, A \cap B$ using intervals.

b) Which of these four sets is/are open? Which is/are closed? Justify your answers.

Problem 7. Decide which of the following statements are true and which are false. Prove the true ones and provide a counterexample for the false ones.

a) If $\{x_n\}$ is Cauchy and $\{y_n\}$ is bounded, then $\{x_n y_n\}$ is Cauchy;

b) If $\{x_n\}$ is a sequence of real numbers that satisfies $x_{2^k} - x_{2^{k-1}} \longrightarrow 0$ as $k \longrightarrow \infty$ and $x_n = 0$ for all $n \neq 2^k$, $k \in \mathbb{N}$, then $\{x_n\}$ is Cauchy;

c) If $\{x_n\}$ and $\{y_n\}$ are Cauchy and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $\{x_n / y_n\}$ is Cauchy;

d) If $\{x_n\}$ and $\{y_n\}$ are Cauchy, then $\{1/(|x_n| + |y_n|)\}$ cannot converge to zero.

Problem 8. Prove that $x_n = \sum_{k=1}^n \frac{1}{k^2}$ is Cauchy.

Problem 9. Prove that $x_n = \frac{(-1)^n (n-1)}{n+1}$ does not converge.

Problem 10. Let $x_n = \sum_{k=1}^n \frac{1}{\sqrt{k+1} + \sqrt{k}}$ for $n \in \mathbb{N}$. Does $\{x_n\}$ converge? Justify your answer.

Problem 11. Let w_n satisfy $|w_n| \leqslant n^2$. Determine whether the following sequences are converging or not. If converging find the limit.

$$x_n = \sqrt{n^2 + 3n} - n - 3; \qquad y_n = \frac{e^n - 1}{3^n - 2^n}; \qquad z_n = \frac{w_n^2 + 4w_n + 5}{n^4 + 3n}. \tag{2}$$

You can use the fact that $r^n \longrightarrow 0$ as $n \longrightarrow \infty$ if $|r| < 1$.

Problem 12. Let $x_0 = 7$ and define x_n iteratively through

$$x_{n+1} = \frac{2x_n}{3} - 1. \tag{3}$$

Does $\lim_{n\to\infty} x_n$ exist? If it does, find the limit. Justify your answers.

Problem 13. Let $\{x_n\}$ be a sequence of real numbers. Prove:

a) $\{x_n\}$ is unbounded if and only if there is a subsequence $\{x_{n_k}\}$ satisfying $|x_{n_k}| \to \infty$.

b) $\{x_n\}$ is bounded if and only if $\exists M \in \mathbb{R}$ such that $\limsup_{n\longrightarrow\infty} x_n \leqslant M$ and $\liminf_{n\to\infty} x_n \geqslant -M$.

Problem 14. Let A be a nonempty subset of $\mathbb{R}$. Let $B = 3A := \{3x : x \in A\}$. Derive the relations between $\sup B$, $\inf B$ and $\sup A$, $\inf A$. Justify your answers. Note that you may need to discuss different cases for c and for $\sup A$.

Problem 15. Let $x_n = (-1)^{3n} + \frac{1}{n^2}$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$. Let $A := \{x_n : n \in \mathbb{N}\} = \{x_1, x_2, \ldots\}$.

a) Find $\max A, \sup A, \min A, \inf A$. Justify your answers.

b) Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$. Justify your answers.

Problem 16. Let $f(x)$ be a continuous function and let $c \in \mathbb{R}$. Prove that the pre-image $f^{-1}(c)$ is a closed set.

Problem 17. Find the limits of

$$\lim_{x\longrightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}; \qquad \lim_{x\longrightarrow 0} \frac{\sqrt{1+x} - 1}{x}; \qquad \lim_{x\longrightarrow\infty} \frac{x^3 + 5x + 6}{\sqrt{x^6 + 3x} - 7}. \tag{4}$$

Indicate clearly what property you are using at each step.

Problem 18. Find and prove the limit

$$\lim_{x\longrightarrow -\infty} \left(\sqrt{x^2 + 2x} + x\right). \tag{5}$$

Problem 19. Let $f(x) : (0, 2) \mapsto \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x^2 - 3x + 2} & x \neq 1 \\ c & x = 1 \end{cases}. \tag{6}$$

Find all $c \in \mathbb{R}$ which makes $f(x)$ continuous at $x = 1$. Justify your answer.

Problem 20. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a real function. Prove that $f(x) \longrightarrow 0$ if and only if $|f(x)| \longrightarrow 0$. Is it true that $f(x) \longrightarrow L \neq 0$ if and only if $|f(x)| \longrightarrow |L|$? Justify your answer.

HARDER PROBLEMS

- Problems at this level may or may not appear in the midterm.
- The solutions for these problems may be a bit sketchy. You are discouraged to read the solution before having seriously worked on the problems.

Problem 21. Let $A_n := \left(1, 1 + \frac{3}{n^2}\right)$, $B_n := \left[1, 1 + \frac{3}{n^2}\right]$, $C_n := \left[1, 1 + \frac{3}{n^2}\right)$, $D_n := \left(1, 1 + \frac{3}{n^2}\right]$ for every $n \in \mathbb{N}$.

a) Represent $\cup_{n=1}^{\infty} A_n$, $\cap_{n=1}^{\infty} A_n$, $\cup_{n=1}^{\infty} B_n$, $\cap_{n=1}^{\infty} B_n$ $\cup_{n=1}^{\infty} C_n$, $\cap_{n=1}^{\infty} C_n$ $\cup_{n=1}^{\infty} D_n$, $\cap_{n=1}^{\infty} D_n$ using intervals.

b) Among these eight sets, which is/are open? Which is/are closed? Justify your answers.

Problem 22. Let $\{x_n\}$ be a sequence such that the subsequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{x_{3n}\}$ are convergent. Show that $\{x_n\}$ is convergent.

Problem 23. Let $f(x)$ satisfy: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1, x_2$ satisfying $x_1, x_2 \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$, $|f(x_1) - f(x_2)| < \varepsilon$. Prove that $\lim_{x \to x_0} f(x)$ exists.

Problem 24. Let $E \subseteq \mathbb{R}$ be nonempty. For every $x \in \mathbb{R}$, its distance to E is defined as

$$d(x) := \inf_{y \in E} |x - y|. \tag{7}$$

a) Prove that $d(x)$ is a continuous function.

b) Prove that inf can be replaced by min if and only if E is closed.

Problem 25. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be a real function. Define $f^+(x) = \lim_{n \longrightarrow \infty} \sup_{|x-y|<1/n} f(y)$, $f^-(x) = \lim_{n \longrightarrow \infty} \inf_{|x-y|<1/n} f(y)$. Prove that f is continuous at x_0 if and only if $f^+(x_0) = f^-(x_0) = f(x_0)$.

Problem 26. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be a real function. Prove that f is continuous if and only if for every open set $A \subseteq \mathbb{R}$, the pre-image $f^{-1}(A)$ is open.

Problem 27. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be a real function satisfying $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that for every closed interval $[a, b]$ with $a, b \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) > \delta$ for all $x \in [a, b]$. Is the claim still true if one or both of a, b is infinity?

Problem 28. (Cesaro average) Let $\{x_n\}$ be a real sequence. Set $y_n = (x_1 + \cdots + x_n)/n$. Show that if $x_n \longrightarrow a \in \mathbb{R}$, then $y_n \longrightarrow a \in \mathbb{R}$. What about the converse, that is does $y_n \longrightarrow a$ guarantees $x_n \longrightarrow a$?

Problem 29. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n}. \tag{8}$$

Use this to prove that if $\lim \frac{x_{n+1}}{x_n}$ exists, so does $\lim (x_n)^{1/n}$. What about the converse?