Math 314 Fall 2012 Final Practice

- You should also
  - review homework problems.
  - try the 2011 final (and you should feel most of its problems are easy).
- Most problems in the final will be at the "Basic" and "Intermediate" levels (First 36 problems).

Basic

Problem 1. Let

$$f(x)=\begin{cases} -1 & x\leqslant -1 \\ a\,x^2+b\,x+c & |x|<1, x\neq 0 \\ 0 & x=0 \\ 1 & x\geqslant 1 \end{cases}. \tag{1}$$

Find $a,b,c\in\mathbb{R}$ such that $f(x)$ is continuous at every x.

Solution. We know that -1, $a\,x^2+b\,x+c$, 0, 1 are all continuous functions, therefore for $f(x)$ to be continuous, we only need to make sure $f(x)$ is continuous at $1,0,-1$.

- At -1. We need

$$-1=a\,(-1)^2+b\,(-1)+c\Longleftrightarrow a-b+c=-1; \tag{2}$$

- At 1. We need

$$a+b+c=1; \tag{3}$$

- At 0. We need

$$c=0. \tag{4}$$

Putting all these together we have

$$a=0, b=1, c=0. \tag{5}$$

Problem 2. Calculate the derivatives of the following functions.

$$f_1(x)=\left(\frac{1+x^2}{1-x^2}\right)^3; \qquad f_2(x)=\sqrt{1+x+x^2}; \qquad f_3(x)=\exp\,[x\ln x]. \tag{6}$$

Solution. We have

$$f_1'(x)=\frac{12\,x\,(x^2+1)^2}{(x^2-1)^4}; \qquad f_2'(x)=\frac{2\,x+1}{2\,\sqrt{x^2+x+1}}; \qquad f_3'(x)=e^{x\ln x}\,[\ln x+1]. \tag{7}$$

Problem 3. Calculate the following limits.

$$\lim_{x\longrightarrow 0}\frac{1-\cos^2 x}{\sqrt{1+x^2}-1}; \qquad \lim_{x\longrightarrow 0}\frac{e^x-e^{-x}-2\,x}{x-\sin x}; \qquad \lim_{x\longrightarrow\infty}\frac{\pi-\arctan x}{\sin\,(1/x)}. \tag{8}$$

Solution.

- We first check that

$$\lim_{x\longrightarrow 0}(1-\cos^2 x)=\lim_{x\longrightarrow 0}\left(\sqrt{1+x^2}-1\right)=0 \tag{9}$$

so we should apply L'Hospital's rule.

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{1-\cos^2 x}{\sqrt{1+x^2}-1} &= \lim_{x\longrightarrow 0}\frac{2\cos x\sin x}{x/\sqrt{1+x^2}}\\ &= \lim_{x\longrightarrow 0}\frac{2\cos x}{\sqrt{1+x^2}}\cdot\frac{\sin x}{x}.\end{aligned} \tag{10}$$

Notice that $\lim_{x\longrightarrow 0}\frac{2\cos x}{\sqrt{1+x^2}}=\frac{2}{1}=2$. We only need to find $\lim_{x\longrightarrow 0}\frac{\sin x}{x}$. Applying L'Hospital's rule again:

$$\lim_{x\longrightarrow 0}\frac{\sin x}{x}=\lim_{x\longrightarrow 0}\frac{\cos x}{1}=1. \tag{11}$$

So finally we conclude

$$\lim_{x\longrightarrow 0}\frac{1-\cos^2 x}{\sqrt{1+x^2}-1}=2. \tag{12}$$

- We first check that

$$\lim_{x\longrightarrow 0}(e^x-e^{-x}-2x)=\lim_{x\longrightarrow 0}(x-\sin x)=0 \tag{13}$$

so L'Hospital's rule can be applied:

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{e^x-e^{-x}-2x}{x-\sin x} &= \lim_{x\longrightarrow 0}\frac{e^x+e^{-x}-2}{1-\cos x}\\ &= \lim_{x\longrightarrow 0}\frac{e^x-e^{-x}}{\sin x}\\ &= \lim_{x\longrightarrow 0}\frac{e^x+e^{-x}}{\cos x}=2.\end{aligned} \tag{14}$$

- We notice that

$$\lim_{x\longrightarrow\infty}(\pi-\arctan x)=\frac{\pi}{2}, \qquad \lim_{x\longrightarrow\infty}\sin\frac{1}{x}=0. \tag{15}$$

To decide whether the limit is ∞ or $-\infty$, we notice $\sin(1/x)>0$ for all $x>1/\pi$. Therefore

$$\lim_{x\longrightarrow\infty}\frac{\pi-\arctan x}{\sin(1/x)}=\infty. \tag{16}$$

Problem 4. Calculate Taylor polynomial to degree 2 with Lagrange form of remainder.

$$f(x)=x\sin(\ln x); \qquad x_0=1. \tag{17}$$

Solution. We have

$$f(1)=0; \tag{18}$$

$$f'(x)=\sin(\ln x)+\cos(\ln x)\Longrightarrow f'(1)=1; \tag{19}$$

$$f''(x)=\frac{1}{x}\cos(\ln x)-\frac{1}{x}\sin(\ln x)\Longrightarrow f''(1)=1; \tag{20}$$

$$f'''(x)=-\frac{\cos(\ln x)-\sin(\ln x)}{x^2}-\frac{\sin(\ln x)+\cos(\ln x)}{x^2}=-\frac{2\cos(\ln x)}{x^2}. \tag{21}$$

Therefore the Taylor polynomial with Lagrange form of remainder is

$$x\sin(\ln x) = (x-1) + \frac{(x-1)^2}{2} - \frac{\cos(\ln\xi)}{3\,\xi^2}(x-1)^3 \tag{22}$$

where ξ is between 1 and x.

Problem 5. Let $f(x) = 2\,x - \sin\,x$ defined on $\mathbb{R}$. Prove that its inverse function g exists and is differentiable. Then calculate $g'(0)$, $g'(\pi-1)$.

Solution. We have $f'(x) = 2 - \cos x \geqslant 1 > 0$ so g exists and is differentiable. We have

$$g'(y) = 1/f'(x) = \frac{1}{2-\cos x} \tag{23}$$

so all we need to do is to figure out x_1, x_2 such that $f(x_1) = 0$ and $f(x_2) = \pi - 1$. It's easily seen that $x_1 = 0, x_2 = \pi/2$. Therefore

$$g'(0) = 1, \qquad g'(\pi-1) = \frac{1}{2}. \tag{24}$$

Problem 6. Which of the following functions is/are differentiable at $x_0 = 0$? Justify your answers

$$f_1(x) = \begin{cases} x+2 & x>0 \\ x-2 & x\leqslant 0 \end{cases}; \qquad f_2(x) = \begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}; \qquad f_3(x) = \begin{cases} x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}. \tag{25}$$

Solution.

- $f_1(x)$. Clearly $f_1(x)$ is not continuous at 0 so is not differentiable there.
- $f_2(x)$. We check

$$\frac{f_2(x) - f_2(0)}{x-0} = \sin\frac{1}{x}. \tag{26}$$

  As the limit $\lim_{x\longrightarrow 0}\sin\frac{1}{x}$ does not exist, $f_2(x)$ is not differentiable at $x_0 = 0$.

- $f_3(x)$. We have

$$\lim_{x\longrightarrow 0}\frac{f_3(x) - f_3(0)}{x-0} = \lim_{x\longrightarrow 0} x\sin\frac{1}{x} = 0 \tag{27}$$

  so $f_3(x)$ is differentiable at $x_0 = 0$.

Problem 7. Let $f(x){:}\,\mathbb{R}\mapsto\mathbb{R}$ be continuous and $x_0\in E$. Define $F(x) := \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & x\neq x_0 \\ c & x = x_0 \end{cases}$. Prove that f is differentiable at x_0 if and only if there is $c\in\mathbb{R}$ such that $F(x)$ is continuous for all $x\in\mathbb{R}$.

Proof. It is clear that $F(x)$ is continuous at all $x\neq x_0$ no matter what c is.

- Only if. If f is differentiable at x_0 then by definition

$$\lim_{x\longrightarrow x_0} F(x) = \lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \tag{28}$$

  So if we set $c = f'(x_0)$, $F(x)$ is also continuous at x_0.

- If. Since $F(x)$ is continuous at x_0, we have

$$c = \lim_{x\longrightarrow x_0} F(x) = \lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0} \tag{29}$$

which by definition means f is differentiable at x_0. □

Problem 8. Calculate the following integrals:

$$I_1 = \int_e^{e^2} \frac{\mathrm{d}x}{x\,(\ln x)^4}; \qquad I_2 = \int_0^4 e^{-\sqrt{x}}\,\mathrm{d}x; \qquad I_3 = \int_1^e x^3 \ln x\,\mathrm{d}x \tag{30}$$

Solution.

- I_1. Change of variable: $y = u(x) = \ln x$. Then we have

$$\begin{aligned} I_1 = \int_e^{e^2} \frac{\mathrm{d}x}{x\,(\ln x)^4} &= \int_e^{e^2} \left(\frac{1}{u(x)^4}\right) u'(x)\,\mathrm{d}x \\ &= \int_{u(e)}^{u(e^2)} \frac{1}{y^4}\,\mathrm{d}y \\ &= \int_1^2 \frac{1}{y^4}\,\mathrm{d}y \\ &= -\frac{1}{3}\,y^{-3}|_1^2 \\ &= \frac{7}{24}. \end{aligned} \tag{31}$$

- I_2. Change of variable: $y = u(x) = \sqrt{x}$. We have

$$\begin{aligned} I_2 = \int_0^4 e^{-\sqrt{x}}\,\mathrm{d}x &= \int_0^4 e^{-u(x)}\,u'(x)\,(2\,u(x))\,\mathrm{d}x \\ &= \int_{u(0)}^{u(4)} e^{-y}\,2\,y\,\mathrm{d}y \\ &= 2\int_0^2 y\,e^{-y}\,\mathrm{d}y \\ &= 2\int_0^2 y\,(-e^{-y})'\,\mathrm{d}y \\ &= 2\left[(-y\,e^{-y})|_0^2 + \int_0^2 e^{-y}\,\mathrm{d}y\right] \\ &= 2\,[-2\,e^{-2} + 1 - e^{-2}] \\ &= 2 - 6\,e^{-2}. \end{aligned} \tag{32}$$

- I_3. We integrate by parts:

$$\begin{aligned} I_3 = \int_1^e x^3 \ln x\,\mathrm{d}x &= \int_1^e \ln x \left(\frac{x^4}{4}\right)'\mathrm{d}x \\ &= \left[\ln x\left(\frac{x^4}{4}\right)\right]_{x=1}^{x=e} - \int_1^e \frac{x^4}{4}\,(\ln x)'\,\mathrm{d}x \\ &= \frac{e^4}{4} - \frac{1}{4}\int_1^e x^3\,\mathrm{d}x \\ &= \frac{3\,e^4+1}{16}. \end{aligned} \tag{33}$$

Problem 9. Prove that the following improper integrals exist and calculate their values:

$$J_1 = \int_0^\infty e^{-2x}\cos(3\,x)\,\mathrm{d}x; \qquad J_2 = \int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}; \qquad J_3 = \int_0^1 (\ln x)^2\,\mathrm{d}x \tag{34}$$

Solution.

- J_1. Notice that $e^{-2x}\cos(3x)$ is continuous on $[0,c]$ for every $c>0$ and is therefore integrable there, we check

$$\begin{aligned}\int_0^c e^{-2x}\cos(3x)\,\mathrm{d}x &= \int_0^c e^{-2x}\left(\frac{1}{3}\sin(3x)\right)'\mathrm{d}x\\ &= e^{-2c}\frac{1}{3}\sin(3c)-e^{-2\cdot 0}\frac{1}{3}\sin(3\cdot 0)\\ &\quad -\int_0^c \frac{1}{3}\sin(3x)\,(e^{-2x})'\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3c)+\frac{2}{3}\int_0^c e^{-2x}\sin(3x)\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3c)-\frac{2}{9}\int_0^c e^{-2x}(\cos(3x))'\,\mathrm{d}x\\ &= \frac{1}{3}e^{-2c}\sin(3c)-\frac{2}{9}\left[e^{-2c}\cos(3c)-e^{-2\cdot 0}\cos(3\cdot 0)+2\int_0^c e^{-2x}\cos(3x)\,\mathrm{d}x\right]\\ &= \frac{1}{3}e^{-2c}\sin(3c)-\frac{2}{9}e^{-2c}\cos(3c)+\frac{2}{9}-\frac{4}{9}\int_0^c e^{-2x}\cos(3x)\,\mathrm{d}x. \qquad (35)\end{aligned}$$

Thus

$$\int_0^c e^{-2x}\cos(3x)\,\mathrm{d}x=\frac{9}{13}\left[\frac{1}{3}e^{-2c}\sin(3c)-\frac{2}{9}e^{-2c}\cos(3c)+\frac{2}{9}\right]. \qquad (36)$$

Taking limit $c\longrightarrow\infty$ we have

$$\lim_{c\longrightarrow\infty}\int_0^c e^{-2x}\cos(3x)\,\mathrm{d}x=\frac{2}{13} \qquad (37)$$

exists and is finite. So the improper integral exists,

$$\int_0^\infty e^{-2x}\cos(3x)\,\mathrm{d}x=\frac{2}{13}. \qquad (38)$$

- J_2. Notice that $\frac{1}{\sqrt{1-x^2}}$ becomes unbounded at $x=1$ and $x=-1$. So we consider

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{1-x^2}} \qquad (39)$$

with $-1<a<b<1$. We apply change of variable $x=\sin y$ with $y\in(\arcsin a,\arcsin b)$. Then $\mathrm{d}x=\cos y\,\mathrm{d}y$ and the integral becomes (note that for the above y we have $\cos y>0$)

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{1-x^2}}=\int_{\arcsin a}^{\arcsin b}\frac{\cos y\,\mathrm{d}y}{\cos y}=\arcsin b-\arcsin a. \qquad (40)$$

Now taking limits $a\longrightarrow -1+, b\longrightarrow 1-$, we have

$$\lim_{a\longrightarrow -1+}\left[\lim_{b\longrightarrow 1-}\int_a^b \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right]=\pi \qquad (41)$$

exists and is finite. So

$$\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}=\pi. \qquad (42)$$

- J_3. As $(\ln x)^2$ is continuous and thus integrable on $[c,1]$ for any $c \in (0,1)$, we consider

$$\begin{aligned}\int_c^1 (\ln x)^2 \,\mathrm{d}x &= [x(\ln x)^2]_c^1 - \int_c^1 2\ln x \,\mathrm{d}x \\ &= -c(\ln c)^2 - 2\left[1\ln 1 - c\ln c - \int_c^1 \mathrm{d}x\right] \\ &= -c(\ln c)^2 + c\ln c + 2(1-c). \end{aligned} \tag{43}$$

Thus

$$\lim_{c \longrightarrow 0+} \int_c^1 (\ln x)^2 \,\mathrm{d}x = 2 \tag{44}$$

exists and is finite, so

$$\int_0^1 (\ln x)^2 \,\mathrm{d}x = 2. \tag{45}$$

Problem 10. Prove by definition that $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ is integrable over $[-1,1]$ and find the value of $\int_{-1}^1 f(x)\,\mathrm{d}x$.

Proof. Let P be any partition of $[-1,1]$. Then we have, since $f(x) \geqslant 0$,

$$L(f,P) = \sum_{i=1}^n \left(\inf_{x \in [x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) \geqslant 0; \tag{46}$$

On the other hand, take $P_n = \left\{x_0 = -1, x_1 = -1 + \frac{1}{n}, \ldots, x_{2n-1} = 1 - \frac{1}{n}, x_{2n} = 1\right\}$. We see that

$$\sup_{[x_{i-1},x_i]} f(x) = \begin{cases} 1 & i = n, n+1 \\ 0 & \text{all other } i \end{cases}. \tag{47}$$

Therefore

$$U(f,P_n) = \sum_{i=1}^{2n} \left(\sup_{x \in [x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) = \sum_{i=n}^{n+1} (x_i - x_{i-1}) = \frac{2}{n}. \tag{48}$$

By definition we have

$$U(f) \leqslant U(f,P_n) = \frac{2}{n} \text{ for all } n \in \mathbb{N} \Longrightarrow U(f) \leqslant 0. \tag{49}$$

This gives $0 \geqslant U(f) \geqslant L(f) \geqslant 0$ which means $U(f) = L(f) = 0$. So $f(x)$ is integrable with $\int_{-1}^1 f(x)\,\mathrm{d}x = 0$. □

Problem 11. (USTC) Is the following calculation correct? Justify your answer.

$$\int_0^\pi \cos^2 x \,\mathrm{d}x = \int_0^0 \frac{\mathrm{d}t}{(1+t^2)^2} = 0 \tag{50}$$

where the change of variable is $t = \tan x$.

Solution. No. Since $\cos^2 x > \frac{1}{2}$ when $x \in (0, \pi/4)$ we have

$$\int_0^\pi \cos^2 x \,\mathrm{d}x \geqslant \int_0^{\pi/4} \cos^2 x \,\mathrm{d}x > \int_0^{\pi/4} \frac{1}{2}\,\mathrm{d}x = \frac{\pi}{8} > 0 \tag{51}$$

so the calculation is not correct. The problem is $u(x) = \tan x$ is not differentiable over $(0,\pi)$.

Problem 12. Let $F(x) := \int_{\sin x}^{x^2+2} e^t \,\mathrm{d}t$. Calculate $F'(x)$ and $F''(x)$.

Solution. Let $G(x) := \int_0^x e^t\,\mathrm{d}t$. Then we have $G'(x) = e^x$, and

$$F(x) = \int_0^{x^2+2} e^t\,\mathrm{d}t + \int_{\sin x}^0 e^t\,\mathrm{d}t = \int_0^{x^2+2} e^t\,\mathrm{d}t - \int_0^{\sin x} e^t\,\mathrm{d}t = G(x^2+2) - G(\sin x). \tag{52}$$

This gives

$$F'(x) = G'(x^2+2)\,(x^2+2)' - G'(\sin x)\,(\sin x)' = 2\,x\,e^{x^2+2} - e^{\sin x}\cos x. \tag{53}$$

Taking derivative again we have

$$F''(x) = (4\,x^2+2)\,e^{x^2+2} + [\sin x - (\cos x)^2]\,e^{\sin x}. \tag{54}$$

Problem 13. Prove the convergence/divergence of (can use convergence/divergence of $\sum n^a$ and $\sum r^n$).

$$\sum_{n=1}^{\infty} \frac{2^n+n}{3^n+5\,n+4}, \qquad \sum_{n=1}^{\infty} \frac{n^2+n}{n^5-4}, \qquad \sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{(n^{1/3}+19)^5}. \tag{55}$$

Proof.

- For all $n \geqslant 1$ we have $2^n > n$. Therefore

  $$\left|\frac{2^n+n}{3^n+5\,n+4}\right| \leqslant \frac{2\cdot 2^n}{3^n} = 2\left(\frac{2}{3}\right)^n. \tag{56}$$

  As $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, so does $\sum_{n=1}^{\infty}\frac{2^n+n}{3^n+5\,n+4}$.

- When $n \geqslant 2$ we have $\frac{n^5}{2} \geqslant 4$. This gives

  $$\left|\frac{n^2+n}{n^5-4}\right| \leqslant \frac{2\,n^2}{n^5/2} = 4\,n^{-3} \tag{57}$$

  when $n \geqslant 2$. As $\sum_{n=1}^{\infty} n^{-3}$ converges, so does $\sum_{n=1}^{\infty}\frac{n^2+n}{n^5-4}$.

- Intuitively when n is large, we have

  $$\frac{\sqrt{n^2+1}}{(n^{1/3}+19)^5} \sim \frac{n}{n^{5/3}} = n^{-2/3}. \tag{58}$$

  So we expect the series to diverge.

  To justify, note that when $n > 19^3$, $n^{1/3} > 19$ and $n^2 > 1$. Therefore for such n we have

  $$\frac{\sqrt{n^2+1}}{(n^{1/3}+19)^5} > \frac{\sqrt{2\,n^2}}{(2\,n^{1/3})^5} = \frac{\sqrt{2}}{32}\,|n^{-2/3}|. \tag{59}$$

  The divergence of $\sum_{n=1}^{\infty} n^{-2/3}$ now implies the divergence of $\sum_{n=1}^{\infty}\frac{\sqrt{n^2+1}}{(n^{1/3}+19)^5}$. □

Problem 14. Prove that $\sum_{n=1}^{\infty}\frac{1}{n\,(n+3)}$ converges and find its sum.

Proof. Since for all $n \in \mathbb{N}$ we have $\frac{1}{n\,(n+3)} \leqslant \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}\frac{1}{n\,(n+3)}$ converges too. To find the sum, we notice

$$S_n = \sum_{k=1}^{n}\frac{1}{k\,(k+3)} = \frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+3}\right) = \frac{1}{3}\left[\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=4}^{n+3}\frac{1}{k}\right] = \frac{1}{3}\left[\sum_{k=1}^{3}\frac{1}{k} - \sum_{k=n+1}^{n+3}\frac{1}{k}\right]. \tag{60}$$

Taking limit $n \longrightarrow \infty$ now gives $S_n \longrightarrow \frac{11}{18}$. □

Problem 15. Prove: If $\sum_{n=1}^{\infty} a_n^2$ converges then $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges. (Hint: $\frac{a^2+b^2}{2} \geqslant a\,b$)

Proof. It suffices to show the convergence of $\sum_{n=1}^{\infty}\left|\frac{a_n}{n}\right|$. Since this is a non-negative series, all we need to show is that it is bounded from above. Notice that

$$\left|\frac{a_n}{n}\right| = |a_n|\frac{1}{n} \leqslant \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right). \tag{61}$$

We know that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are convergent, therefore

$$S_n = \sum_{k=1}^{n}\left|\frac{a_k}{k}\right| \leqslant \frac{1}{2}\left[\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n}\frac{1}{n^2}\right] < \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty}\frac{1}{n^2}\right] \in \mathbb{R}. \tag{62}$$

So $\sum_{n=1}^{\infty}\left|\frac{a_n}{n}\right|$ is bounded above and thus converges. □

Problem 16. Study the convergence/divergence of

$$\sum_{n=1}^{\infty}\frac{1}{n^{(1+1/n)}}. \tag{63}$$

Solution. It diverges because $n^{1/n} \leqslant 2$. To prove this statement, we only need to prove $2^n \geqslant n$ for all $n \in \mathbb{N}$. Use mathematical induction: The statement $P(n) =$ "$2^n \geqslant n$".

- $P(1)$ is true. We have $2^1 \geqslant 1$.
- If $P(n)$ is true, that is $2^n \geqslant n$, then we have

  $$2^{n+1} = 2\cdot 2^n \geqslant 2n \geqslant n+1 \tag{64}$$

  therefore $P(n+1)$ is true.

Now we have

$$\frac{1}{n^{(1+1/n)}} \geqslant \frac{1}{2}\frac{1}{n}. \tag{65}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty}\frac{1}{n^{(1+1/n)}}$.

Problem 17. Prove that

$$\sum_{n=1}^{\infty}\frac{1}{2n+1} = \infty. \tag{66}$$

You can use the divergence of $\sum_{n=1}^{\infty}\frac{1}{n}$.

Proof. For all $n \geqslant 1$ we have

$$\frac{1}{2n+1} \geqslant \frac{1}{3}\frac{1}{n}. \tag{67}$$

Therefore $\sum_{n=1}^{\infty}\frac{1}{n} = \infty \Longrightarrow \sum_{n=1}^{\infty}\frac{1}{2n+1} = \infty$. □

Problem 18. Consider $\sum_{n=1}^{\infty} n\,r^n$. Identify the values of $r \in \mathbb{R}$ such that it is convergent. Justify your answer. You can use the fact that $\lim_{n\longrightarrow\infty} n\,r^n = 0$ when $|r| < 1$.

Solution.

Apply ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{n}|r| \longrightarrow |r| \text{ as } n \longrightarrow \infty. \tag{68}$$

Therefore the series converges for $|r| < 1$. On the other hand, when $|r| \geqslant 1$ it is clear that $\lim_{n\longrightarrow\infty} n\,r^n = 0$ does not hold. Therefore the series is divergent for such r.

Intermediate

Problem 19. Let f, g be continuous at $x_0 \in \mathbb{R}$. Then so are

$$F(x) := \max\{f(x), g(x)\}, \qquad G(x) := \min\{f(x), g(x)\}. \tag{69}$$

Proof. Note that we have

$$F(x) = \max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2} \tag{70}$$

and

$$G(x) = \min\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2} \tag{71}$$

As f, g are continuous at x_0, so are $\frac{f(x)+g(x)}{2}$ and $\frac{|f(x)-g(x)|}{2}$. Consequently F, G are also continuous at x_0. □

Problem 20. Prove the following.

a) There is **exactly one** $x \in (0, 1)$ such that

$$x^{1/2} e^x = 1. \tag{72}$$

b) There are infinitely many $x \in \mathbb{R}$ satisfying

$$x \sin x = 1. \tag{73}$$

Proof.

a) Since $x^{1/2}$ and e^x are both continuous on $\mathbb{R}$, $x^{1/2} e^x$ is also continuous on $\mathbb{R}$. We try to use intermediate value theorem. Denote $f(x) = x^{1/2} e^x$. Calculate

$$f(0) = 0, \qquad f(1) = e \Longrightarrow 1 \in (f(0), f(1)). \tag{74}$$

Thus there is $\xi \in (0, 1)$ such that $f(\xi) = 1$.

To show that ξ is the only solution to the equation, we check that $f(x)$ is strictly increasing:

$$f'(x) = \left(\frac{1}{2} x^{-1/2} + x^{1/2}\right) e^x > 0 \tag{75}$$

for all $x \in (0, 1)$. Thus $f(x) > 1$ when $x > \xi$ and $f(x) < 1$ when $x < \xi$.

b) Since $x, \sin x$ are both continuous on $\mathbb{R}$, $f(x) := x \sin x$ is also continuous on $\mathbb{R}$. Now we check, for every $n \in \mathbb{N}$,

$$f(n\pi) = 0 < 1, \qquad f\left(n\pi + \frac{\pi}{2}\right) = n\pi + \frac{\pi}{2} > 1. \tag{76}$$

Thus the intermediate value theorem gives the existence of $x_n \in \left(n\pi, n\pi + \frac{\pi}{2}\right)$ satisfying $f(x_n) = 1$. So there are infinitely many solutions to $f(x) = 1$. □

Problem 21. Let $f(x)$ be differentiable at x_0 with derivative $f'(x_0) = 3$. Calculate

$$\lim_{n \longrightarrow \infty} (3n^2 + 2n - 1)\left[f\left(x_0 + \frac{2}{n^2}\right) - f(x_0)\right] \tag{77}$$

Solution. Since $f(x)$ is differentiable at x_0, we have

$$\lim_{n \longrightarrow \infty} \frac{n^2}{2}\left[f\left(x_0 + \frac{2}{n^2}\right) - f(x_0)\right] = f'(x_0) = 3. \tag{78}$$

Therefore as $n \longrightarrow \infty$.

$$\begin{aligned}(3n^2 + 2n - 1)\left[f\left(x_0 + \frac{2}{n^2}\right) - f(x_0)\right] &= \left(6 + \frac{4}{n} - \frac{2}{n^2}\right)\left\{\frac{n^2}{2}\left[f\left(x_0 + \frac{2}{n^2}\right) - f(x_0)\right]\right\} \\ &\longrightarrow 6 f'(x_0) = 18.\end{aligned} \tag{79}$$

Problem 22. Let f, g be differentiable on (a, b) and continuous on $[a, b]$. Further assume $f(a) = g(b)$, $f(b) = g(a)$. Prove that there is $\xi \in (a, b)$ such that $f'(\xi) = -g'(\xi)$.

Proof. Let $h(x) := f(x) + g(x)$. Then we have $h(x)$ differentiable on (a, b) and continuous on $[a, b]$, and furthermore

$$h(a) = f(a) + g(a) = f(a) + f(b) = g(b) + f(b) = h(b). \tag{80}$$

Applying the mean value theorem we have: there is $\xi \in (a, b)$ such that $h'(\xi) = 0$. But this is exactly $f'(\xi) = -g'(\xi)$. □

Problem 23. Prove the following inequalities

a) $|\cos x - \cos y| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$;
b) $|\arctan x - \arctan y| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$;
c) $\frac{a-b}{a} < \ln \frac{a}{b} < \frac{a-b}{b}$, $0 < b < a$.

Proof.

a) By mean value theorem

$$|\cos x - \cos y| = |(\sin \xi)\,(x - y)| = |\sin \xi|\,|x - y| \leqslant |x - y|. \tag{81}$$

b) By mean value theorem

$$|\arctan x - \arctan y| = \left| \frac{1}{1 + \xi^2}\,(x - y) \right| = \left| \frac{1}{1 + \xi^2} \right| |x - y| \leqslant |x - y|. \tag{82}$$

c) By mean value theorem

$$\ln \frac{a}{b} = \ln a - \ln b = \frac{1}{\xi}\,(a - b). \tag{83}$$

Since $b < \xi < a$ and $a - b > 0$, we have

$$\frac{a - b}{a} < \frac{a - b}{\xi} < \frac{a - b}{b}. \tag{84}$$

□

Problem 24.

a) Let $a \in (0, 1)$. Prove that

$$\lim_{n \longrightarrow \infty} [(n + 1)^a - n^a] = 0. \tag{85}$$

You can use $(x^a)' = a\, x^{a-1}$.

b) Prove that

$$\lim_{n \longrightarrow \infty} \left[\sin\left((n + 1)^{1/3} \right) - \sin\left(n^{1/3} \right) \right] = 0. \tag{86}$$

Proof.

a) Applying Mean Value Theorem to $f(x) = x^a$, we have

$$0 \leqslant (n + 1)^a - n^a = a\, \xi^{a-1} [(n + 1) - n] = \frac{a}{\xi^{1-a}} \leqslant \frac{1}{n^{1-a}} \tag{87}$$

where the last inequality follows from $\xi \in (n, n + 1)$ and $1 - a > 0$. Now take $n \longrightarrow \infty$, Squeeze Theorem gives

$$\lim_{n \longrightarrow \infty} [(n + 1)^a - n^a] = 0. \tag{88}$$

b) By Mean Value Theorem we have

$$\sin\left((n + 1)^{1/3} \right) - \sin\left(n^{1/3} \right) = \cos(\xi) \left[(n + 1)^{1/3} - n^{1/3} \right] \tag{89}$$

where $\xi \in \left(n^{1/3}, (n+1)^{1/3}\right)$. This gives

$$\left|\sin\left((n+1)^{1/3}\right) - \sin\left(n^{1/3}\right)\right| \leqslant (n+1)^{1/3} - n^{1/3}. \tag{90}$$

Thanks to a) we have $\lim_{n\longrightarrow\infty}\left[(n+1)^{1/3} - n^{1/3}\right] = 0$. Application of Squeeze Theorem to

$$-\left[(n+1)^{1/3} - n^{1/3}\right] \leqslant \sin\left((n+1)^{1/3}\right) - \sin\left(n^{1/3}\right) \leqslant (n+1)^{1/3} - n^{1/3} \tag{91}$$

gives the desired result. □

Problem 25. (USTC) Let f be differentiable on $\mathbb{R}$, $f(0) = 0$ and $f'(x)$ is strictly increasing. Prove that $\frac{f(x)}{x}$ is strictly increasing on $(0, \infty)$.

Proof. We calculate

$$\left(\frac{f(x)}{x}\right)' = \frac{f'(x)\, x - f(x)}{x^2}. \tag{92}$$

Now notice that by the mean value theorem,

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(\xi) \tag{93}$$

for some $\xi \in (0, x)$. As $f'(x)$ is strictly increasing, $f'(\xi) < f'(x)$ therefore

$$f(x) = x\, f'(\xi) < x\, f'(x) \tag{94}$$

thanks to $x > 0$.

This gives

$$\left(\frac{f(x)}{x}\right)' = \frac{f'(x)\, x - f(x)}{x^2} > 0 \tag{95}$$

for all $x \in (0, \infty)$. So $\frac{f(x)}{x}$ is strictly increasing on $(0, \infty)$. □

Problem 26. Let $f(x)$ be differentiable on $(-\infty, 0)$ and $(0, \infty)$. Assume that

$$\lim_{x\longrightarrow 0-} f'(x) = A, \qquad \lim_{x\longrightarrow 0+} f'(x) = B. \tag{96}$$

Prove that if $A \neq B$ then $f(x)$ is not differentiable at $x = 0$.

Proof. First notice that if $f(x)$ is not continuous at $x = 0$ then it is not differentiable there. In the following we assume $f(x)$ is continuous at $x = 0$.

Take one sequence $x_n < 0$, $x_n \longrightarrow 0$ and another sequence $y_n > 0$, $y_n \longrightarrow 0$. Then by Mean Value Theorem (note that we can apply MVT because now $f(x)$ is continuous on the closed intervals $[x_0, 0]$ and $[0, y_n]$) there are $\xi_n \in (x_n, 0)$ and $\eta_n \in (0, y_n)$ such that

$$\frac{f(x_n) - f(0)}{x_n - 0} = f'(\xi_n); \qquad \frac{f(y_n) - f(0)}{y_n - 0} = f'(\eta_n). \tag{97}$$

As $x_n, y_n \longrightarrow 0$, application of Squeeze Theorem gives $\xi_n, \eta_n \longrightarrow 0$. Therefore

$$\lim_{n\longrightarrow\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\longrightarrow\infty} f'(\xi_n) = A \neq B = \lim_{n\longrightarrow\infty} f'(\eta_n) = \lim_{n\longrightarrow\infty} \frac{f(y_n) - f(0)}{y_n - 0} \tag{98}$$

which means

$$\lim_{x\longrightarrow 0} \frac{f(x) - f(0)}{x - 0} \tag{99}$$

does not exist and therefore $f(x)$ is not differentiable at 0. □

Problem 27. Let $a > 1$. Assume $f(x)$ satisfies $|f(x) - f(y)| \leqslant |x - y|^a$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Proof. We show that $f(x)$ is differentiable and $f'(x) = 0$. Take any $x_0 \in \mathbb{R}$, we have

$$-|x - x_0|^{a-1} \leqslant -\frac{|f(x) - f(x_0)|}{|x - x_0|} \leqslant \frac{f(x) - f(x_0)}{x - x_0} \leqslant \frac{|f(x) - f(x_0)|}{|x - x_0|} \leqslant |x - x_0|^{a-1}. \tag{100}$$

Since $a > 1$, $\lim_{x \longrightarrow x_0} |x - x_0|^{a-1} = 0$. Application of Squeeze Theorem gives

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = 0 \Longrightarrow f'(x_0) = 0. \tag{101}$$

Therefore $f'(x) = 0$ for all $x \in \mathbb{R}$ and $f(x)$ is a constant. □

Problem 28. (USTC) Let f, g be differentiable on $[a, \infty)$, and $|f'(x)| \leqslant g'(x)$ for all $x \in [a, \infty)$. Prove that

$$|f(x) - f(a)| \leqslant g(x) - g(a) \tag{102}$$

for all $x > a$. (Hint: Cauchy's generalized mean value theorem.)

Proof. Since $|a| = \max(a, -a)$ for any $a \in \mathbb{R}$, it suffices to prove

$$f(x) - f(a) \leqslant g(x) - g(a) \text{ and } (-f)(x) - (-f)(a) \leqslant g(x) - g(a). \tag{103}$$

It is clear that $g'(x) \geqslant 0$ so $g(x)$ is increasing. Therefore if $f(x) = f(a)$, we have

$$|f(x) - f(a)| = 0 \leqslant g(x) - g(a). \tag{104}$$

Thus in the following we only consider those x such that $f(x) \neq f(a)$. This implies $g(x) > g(a)$.

Applying generalized mean value theorem to f and g we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \tag{105}$$

for some $\xi \in (a, x)$. As $f(x) \neq f(a)$, $f'(\xi) \neq 0$ which means $g'(\xi) \neq 0$.

Since $|f'(\xi)| \leqslant g'(\xi)$ we have $f'(\xi)/g'(\xi) \leqslant 1$ so

$$\frac{f(x) - f(a)}{g(x) - g(a)} \leqslant 1 \xrightarrow{\text{Recall that } g(x) - g(a) > 0} f(x) - f(a) \leqslant g(x) - g(a). \tag{106}$$

On the other hand, applying the same theorem to $-f$ and g gives

$$\frac{-(f(x) - f(a))}{g(x) - g(a)} \leqslant 1 \Longrightarrow -(f(x) - f(a)) \leqslant g(x) - g(a). \tag{107}$$

Combining the two inequalities we reach

$$|f(x) - f(a)| \leqslant g(x) - g(a) \tag{108}$$

as required. □

Problem 29. Let f be continuous and g be integrable on $[a, b]$. Further assume that $g(x)$ doesn't change sign in $[a, b]$. Prove that there is $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = f(\xi) \int_a^b g(x)\, \mathrm{d}x. \tag{109}$$

Does the conclusion still hold if we drop "$g(x)$ doesn't change sign in $[a, b]$"?

Proof. First notice that we only need to prove for the case $g(x) \geqslant 0$ since the case $g(x) \leqslant 0$ can be immediately obtained through the former case by considering $-g(x)$.

Next we notice that if $\int_a^b g(x)\,\mathrm{d}x = 0$, then $\left|\int_a^b f(x)\,g(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,g(x)\,\mathrm{d}x \leqslant A\int_a^b g(x)\,\mathrm{d}x = 0$ where $A = \max_{[a,b]}|f(x)| \in \mathbb{R}$ whose existence follows from the continuity of f on $[a,b]$. Therefore in this case

$$0 = \int_a^b f(x)\,g(x)\,\mathrm{d}x = f(\xi)\int_a^b g(x)\,\mathrm{d}x = 0 \tag{110}$$

holds for any $\xi \in [a,b]$.

Now we assume $\int_a^b g(x)\,\mathrm{d}x > 0$. As $f(x)$ is continuous on $[a,b]$ there are $\xi_1, \xi_2 \in [a,b]$ such that

$$f(\xi_1) \leqslant f(x) \leqslant f(\xi_2) \tag{111}$$

for all $x \in [a,b]$. As $g(x) \geqslant 0$, we have

$$f(\xi_1)\,g(x) \leqslant f(x)\,g(x) \leqslant f(\xi_2)\,g(x) \tag{112}$$

which leads to

$$f(\xi_1)\int_a^b g(x)\,\mathrm{d}x \leqslant \int_a^b f(x)\,g(x)\,\mathrm{d}x \leqslant f(\xi_2)\int_a^b g(x)\,\mathrm{d}x. \tag{113}$$

that is

$$f(\xi_1) \leqslant \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \leqslant f(\xi_2). \tag{114}$$

Application of the Intermediate Value Theorem now gives the existence of $\xi \in [a,b]$ satisfying

$$f(\xi) = \frac{\int_a^b f(x)\,g(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x} \tag{115}$$

which is what we need to prove.

If g changes sign the conclusion does not hold anymore. For example take $g(x) = \sin x$, $f(x) = 1$ and $a = 0, b = 2\pi$. □

Problem 30. Prove the following inequalities:

a) $\int_0^1 e^{-x^2}\,\mathrm{d}x > \int_1^2 e^{-x^2}\,\mathrm{d}x$;

b) $\int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x > \int_0^{\pi/2} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$;

Proof.

a) We do a change of variable: $y = x - 1$ for the second integral:

$$\int_1^2 e^{-x^2}\,\mathrm{d}x = \int_0^1 e^{-(y+1)^2}\,\mathrm{d}y = \int_0^1 e^{-(x+1)^2}\,\mathrm{d}x. \tag{116}$$

Now for $x \in (0,1)$ we have

$$-x^2 > -(x+1)^2 \Longrightarrow e^{-x^2} > e^{-(x+1)^2} \tag{117}$$

which gives

$$\int_0^1 e^{-x^2}\,\mathrm{d}x > \int_0^1 e^{-(x+1)^2}\,\mathrm{d}x \tag{118}$$

as desired.

b) We show that for $x \in \left(0, \frac{\pi}{2}\right)$, $0 \leqslant \frac{\sin x}{x} < 1$. The first inequality is obvious. To show the second, we calculate

$$\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}. \tag{119}$$

Now let $f(x) = x\cos x - \sin x$ and notice that

$$f(0) = 0, \qquad f'(x) = -x\sin x < 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right) \tag{120}$$

therefore

$$f(x) < 0 \tag{121}$$

for all $x > 0$. Consequently $f(x)$ is strictly decreasing which means

$$x \cos x - \sin x = f(x) < f(0) = 0. \tag{122}$$

Therefore

$$\left(\frac{\sin x}{x}\right)' < 0 \Longrightarrow \frac{\sin x}{x} \text{ is strictly decreasing.} \tag{123}$$

As $\lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1$, this means

$$\frac{\sin x}{x} < 1 \tag{124}$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

From this we have

$$\frac{\sin x}{x} > \frac{\sin^2 x}{x^2} \qquad \forall x \in \left(0, \frac{\pi}{2}\right) \tag{125}$$

which gives

$$\int_0^{\pi/2} \frac{\sin x}{x} \, \mathrm{d}x > \int_0^{\pi/2} \frac{\sin^2 x}{x^2} \, \mathrm{d}x. \tag{126}$$

□

Problem 31. (USTC) Prove

$$\int_0^{2\pi} \left[\int_x^{2\pi} \frac{\sin t}{t} \, \mathrm{d}t\right] \mathrm{d}x = 0. \tag{127}$$

(Hint: Set $u(x) = \int_x^{2\pi} \frac{\sin t}{t} \, \mathrm{d}t$ then integrate by parts)

Proof. Set

$$u(x) = \int_x^{2\pi} \frac{\sin t}{t} \, \mathrm{d}t, \qquad v(x) = x \tag{128}$$

then we have

$$\begin{aligned} \int_0^{2\pi} \left[\int_x^{2\pi} \frac{\sin t}{t} \, \mathrm{d}t\right] \mathrm{d}x &= \int_0^{2\pi} u(x) \, v'(x) \, \mathrm{d}x \\ &= u(2\pi) \, v(2\pi) - u(0) \, v(0) - \int_0^{2\pi} v(x) \, u'(x) \, \mathrm{d}x \\ &= 0 - 0 - \int_0^{2\pi} x \left(-\frac{\sin x}{x}\right) \mathrm{d}x \\ &= 0. \end{aligned}$$

□

Problem 32. Let f be continuous on $\mathbb{R}$. Let $a, b \in \mathbb{R}, a < b$. Then

$$\lim_{h \longrightarrow 0} \int_a^b \frac{f(x+h) - f(x)}{h} \, \mathrm{d}x = f(b) - f(a). \tag{129}$$

Proof. Let $F(x)$ be an antiderivative of f. Since $f(x)$ is continuous on the closed interval $[a, b]$ it is integrable. We have

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a) \tag{130}$$

On the other hand,

$$\int_a^b f(x+h) \, \mathrm{d}x = \int_{a+h}^{b+h} f(y) \, \mathrm{d}y = F(b+h) - F(a+h). \tag{131}$$

Thus

$$\begin{aligned}\lim_{h\longrightarrow 0}\int_a^b \frac{f(x+h)-f(x)}{h} &= \lim_{h\longrightarrow 0}\frac{1}{h}[(F(b+h)-F(b)-(F(a+h)-F(a)))]\\ &= F'(b)-F'(a)\\ &= f(b)-f(a)\end{aligned} \tag{132}$$

Thanks to FTC Version 2. □

Problem 33. (USTC) Let f be integrable. Prove that

$$\int_0^\pi x\, f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x. \tag{133}$$

(Hint: Change of variable: $t = \pi - x$.)

Proof. Do the change of variable as in the hint, we have

$$\int_0^\pi \left(\frac{\pi}{2}-x\right) f(\sin x)\,\mathrm{d}x = \int_\pi^0 \left(t-\frac{\pi}{2}\right) f(\sin t)\,(-1)\,\mathrm{d}t = -\int_0^\pi \left(\frac{\pi}{2}-t\right) f(\sin t)\,\mathrm{d}t \tag{134}$$

That is

$$\int_0^\pi \left(\frac{\pi}{2}-x\right) f(\sin x)\,\mathrm{d}x = -\int_0^\pi \left(\frac{\pi}{2}-x\right) f(\sin x)\,\mathrm{d}x \tag{135}$$

so

$$\int_0^\pi \left(\frac{\pi}{2}-x\right) f(\sin x)\,\mathrm{d}x = 0. \tag{136}$$

□

Problem 34. Apply Ratio/Root tests to determine the convergence/divergence of the following series (You need to decide which one is more convenient to use).

$$\sum_{n=1}^\infty \frac{1}{2^n}(1+1/n)^{n^2}; \qquad \sum_{n=1}^\infty (n!)\,x^n; \qquad \sum_{n=1}^\infty \frac{(n!)}{n^n}\,x^n. \tag{137}$$

You can use the fact $(1+1/n)^n \longrightarrow e$, and the Stirling's formula

$$\lim_{n\longrightarrow\infty}\frac{n!}{\sqrt{2\,\pi\,n}\,(n/e)^n} = 1 \tag{138}$$

without proof.

Solution.

- We apply root test:

$$|a_n|^{1/n} = \frac{1}{2}(1+1/n)^n. \tag{139}$$

As $\lim_{n\longrightarrow\infty}\frac{1}{2}(1+1/n)^n = \frac{e}{2}$, we have

$$\liminf_{n\longrightarrow\infty}|a_n|^{1/n} = \frac{e}{2} > 1 \tag{140}$$

so the series diverges.

- We apply ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = (n+1)\,|x|. \tag{141}$$

This leads to

$$\lim_{n\longrightarrow\infty}(n+1)\,|x| = \begin{cases} 0 & x=0\\ \infty & x\neq 0\end{cases}. \tag{142}$$

Since the limit exists, we have

$$\limsup_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 0 < 1 \text{ when } x = 0; \qquad \liminf_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = \infty > 1 \text{ when } x \neq 0. \tag{143}$$

So the series converges for $x = 0$ but diverges for all $x \neq 0$.

- We apply ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)\,|x|}{(n+1)^{n+1}\,n^{-n}} = \frac{|x|}{(1+1/n)^n} \longrightarrow \frac{|x|}{e}. \tag{144}$$

The ratio test then gives convergence when $|x| < e$ and divergence when $|x| > e$. When $|x| = e$, we have

$$|a_n| = \frac{(n!)}{n^n}\,e^n = \frac{n!}{(n/e)^n} \tag{145}$$

and Stirling's formula gives

$$\lim_{n\longrightarrow\infty} \frac{|a_n|}{\sqrt{2\,\pi\,n}} = 1. \tag{146}$$

This means $a_n \not\longrightarrow 0$ so the series diverges.

Summarizing, the series converges when $|x| < e$ and diverges when $|x| \geqslant e$.

Problem 35. $a_n \geqslant 0$, $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n\,a_{n+1}}$ converges. On the other hand, if a_n furthermore is decreasing, then $\sum_{n=1}^{\infty} \sqrt{a_n\,a_{n+1}}$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges. Any example of if a_n is not decreasing then not true? (Take $a_n = 0$ for all n even)

Proof. First we note that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longleftrightarrow \sum_{n=1}^{\infty} a_{n+1} \text{ converges.} \tag{147}$$

since

$$\sum_{k=1}^{n} a_k = a_1 + \sum_{k=1}^{n-1} a_{k+1}. \tag{148}$$

- If $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} a_{n+1}$ and then $\sum_{n=1}^{\infty} (a_n + a_{n+1})$. The convergence of $\sum_{n=1}^{\infty} \sqrt{a_n\,a_{n+1}}$ then follows from

$$\sqrt{a_n\,a_{n+1}} \leqslant \frac{1}{2}\,(a_n + a_{n+1}). \tag{149}$$

- If a_n is decreasing, we have $a_n \geqslant a_{n+1} \Longrightarrow a_{n+1} \leqslant \sqrt{a_n\,a_{n+1}}$. Thus the convergence of $\sum_{n=1}^{\infty} \sqrt{a_n\,a_{n+1}}$ implies the convergence of $\sum_{n=1}^{\infty} a_{n+1}$ and then that of $\sum_{n=1}^{\infty} a_n$.

If a_n is not decreasing then $\sum_{n=1}^{\infty} \sqrt{a_n\,a_{n+1}}$ converges $\not\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges. For example take $a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$. □

Problem 36. Let $a_n \geqslant 0$. Prove that $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges. Is the converse true? Justify your answer.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\longrightarrow\infty} a_n = 0$. Thus there is $N \in \mathbb{N}$ such that for all $n > N$, $a_n < 1$. Now for these n we have

$$|a_n^2| = a_n^2 < a_n. \tag{150}$$

Therefore the convergence of $\sum_{n=1}^{\infty} a_n$ gives the convergence of $\sum_{n=1}^{\infty} a_n$.

The converse is not true. Take $a_n = \frac{1}{n}$. □

Advanced

Problem 37. A function $f(x): E \mapsto \mathbb{R}$ is called "uniformly continuous" if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in E$ satisfying $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

a) Prove that if f is uniformly continuous, then it is continuous.

b) Give an example of a continuous function that is not uniformly continuous. Justify your answer.

c) If $f: E \mapsto \mathbb{R}$ is continuous with E a bounded closed set, then f is uniformly continuous.

d) Prove that if f is continuous on $[a, b]$, then it is integrable on $[a, b]$.

Proof.

a) This is obvious.

b) $f(x) = 1/x$ defined for $x > 0$. Take $\varepsilon = 1$. Then for any $\delta > 0$ we can take $n \in \mathbb{N}$ such that $n > \delta^{-1}$. Then we have $\left|\frac{1}{n} - \frac{1}{n+1}\right| < \delta$ but $|f(1/n) - f(1/(n+1))| = 1 \geqslant \varepsilon$.

c) Assume the contrary. Then there is $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there are x_n, y_n such that

$$|x_n - y_n| < 1/n, \qquad |f(x_n) - f(y_n)| \geqslant \varepsilon_0. \tag{151}$$

Applying Bolzano-Weierstrass, there is a subsequence $x_{n_k} \longrightarrow \xi \in [a, b]$. As $|x_n - y_n| \longrightarrow 0$, we have $y_{n_k} \longrightarrow \xi$ too. But then $|\lim_{k \longrightarrow \infty} f(x_{n_k}) - \lim_{k \longrightarrow \infty} f(y_{n_k})| \geqslant \varepsilon_0$, contradicting the continuity of f.

d) From c) we know that f is uniformly continuous. Now for any $\varepsilon > 0$, take δ such that for all $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon/(b - a)$.

Now take any partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$ with $|x_i - x_{i-1}| < \delta$ for all $i = 1, 2, \ldots, n$. Then we have

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (\sup f - \inf f)\,(x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\varepsilon}{b - a}\,(x_i - x_{i-1}) = \varepsilon. \tag{152}$$

Therefore f is integrable. □

Problem 38. Let $f(x)$ be continuous over $\mathbb{R}$, and satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there is $a \in \mathbb{R}$ such that $f(x) = a\,x$.

Proof. First

$$f(0 + 0) = f(0) + f(0) \Longrightarrow f(0) = 0; \tag{153}$$

Now let $a = f(1)$. Clearly $f(n) = n\,a$. Next consider any rational number $q = \frac{n}{m}$. Then we have

$$n\,a = f(n) = f(m\,q) = m\,f(q) \Longrightarrow f(q) = a\,q. \tag{154}$$

Finally for any $x \in \mathbb{R} \backslash \mathbb{Q}$, there is $q_n \longrightarrow x$. Since $f(x)$ is continuous we have

$$f(x) = \lim_{n \longrightarrow \infty} f(q_n) = \lim_{n \longrightarrow \infty} a\,q_n = a\,x. \tag{155}$$

Thus ends the proof. □

Problem 39. (USTC) Let $f(x)$ be differentiable. Assume that there are $a < b$ such that $f(a) = f(b) = 0$, $f'(a)\,f'(b) > 0$. Prove that there is $\xi \in (a, b)$ such that $f(\xi) = 0$.

Proof. There are two cases, $f'(a) > 0$, $f'(b) > 0$ and $f'(a) < 0$, $f'(b) < 0$. Considering $-f$ instead of f would turn any one case into the other, so we only consider the first case here.

Since $f'(a) > 0$,

$$\lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a} > 0. \tag{156}$$

Thus there is $x_1 \in \left(a, \frac{a+b}{2}\right)$ such that $f(x_1) > 0$; On the other hand, since $f'(b) > 0$

$$\lim_{x \longrightarrow b} \frac{f(x) - f(b)}{x - b} > 0 \tag{157}$$

which means there is $x_2 \in \left(\frac{a+b}{2}, b\right)$ such that $f(x_2) < 0$.

Now $f(x)$ is differentiable on (a, b) so is continuous on (a, b). Application of Intermediate Value Theorem gives the existence of $\xi \in (x_1, x_2) \subseteq (a, b)$ satisfying $f(\xi) = 0$. □

Problem 40. Let $f(x)$ be continuous on (a, b). Assume there is $x_0 \in (a, b)$ such that $f'''(x_0)$ exists. Prove that there are constants A, B, C, D such that

$$\lim_{h \longrightarrow 0} \frac{A\, f(x_0 + h) + B\, f(x_0) + C\, f(x_0 - h) + D\, f(x_0 - 2\,h)}{h^3} = f'''(x_0) \tag{158}$$

and find their values. (Hint: L'Hospital)

Proof. First notice that if $A + B + C + D \neq 0$, then the limit cannot be finite. Therefore we have

$$A + B + C + D = 0. \tag{159}$$

Now apply L'Hospital: Note that since $f'''(x_0)$ exists, $f''(x)$ must exist and be continuous on some (a_2, b_2) containing x_0, then so does $f'(x)$ and $f(x)$. Therefore $f(x_0 + h)$ (and others) is differentiable at $h = 0$.

$$f'''(x_0) = \lim_{h \longrightarrow 0} \frac{A\, f'(x_0 + h) - C\, f'(x_0 - h) - 2\, D\, f'(x_0 - 2\,h)}{3\, h^2}. \tag{160}$$

This gives us

$$A - C - 2\, D = 0. \tag{161}$$

Applying L'Hospital again:

$$f'''(x_0) = \lim_{h \longrightarrow 0} \frac{A\, f''(x_0 + h) + C\, f''(x_0 - h) + 4\, D\, f''(x_0 - 2\,h)}{6\, h} \tag{162}$$

we obtain

$$A + C + 4\, D = 0. \tag{163}$$

Note that we cannot apply L'Hospital's rule anymore since it requires $f'''(x)$ to exist in some open interval around x_0. But we can use definition: (In fact we can use Toy L'Hospital here...)

$$\begin{aligned} \lim_{h \longrightarrow 0} \frac{A\, f''(x_0 + h) + C\, f''(x_0 - h) + 4\, D\, f''(x_0 - 2\,h)}{6\, h} &= \lim_{h \longrightarrow 0} A\, \frac{f''(x_0 + h) - f''(x_0)}{6\, h} \\ &\quad + \lim_{h \longrightarrow 0} C\, \frac{f''(x_0 - h) - f''(x_0)}{6\, h} \\ &\quad + \lim_{h \longrightarrow 0} 4\, D\, \frac{f''(x_0 - 2\,h) - f''(x_0)}{6\, h} \\ &= \frac{A - C - 8\, D}{6}\, f'''(x_0). \end{aligned} \tag{164}$$

(165)

This implies

$$A - C - 8\, D = 6. \tag{166}$$

Putting things together, it is sufficient and necessary for the constants to satisfy

$$\begin{aligned} A+B+C+D &= 0 && (167)\\ A-C-2D &= 0 && (168)\\ A+C+4D &= 0 && (169)\\ A-C-8D &= 6 && (170)\end{aligned}$$

Notice that D can be solved from the 2nd and the 4th equation: $D=-1$. This gives

$$A-C=-2, \qquad A+C=4 \Longrightarrow A=1, C=3. \tag{171}$$

Finally we obtain $B=-3$. Summarizing,

$$\lim_{h\longrightarrow 0} \frac{f(x_0+h)-3\,f(x_0)+3\,f(x_0-h)-f(x_0-2\,h)}{h^3} = f'''(x_0) \tag{172}$$

□

Problem 41. (USTC) Let $f(x)$ be differentiable at x_0 with $f(x_0)\neq 0$ and $f'(x_0)=5$. Take for granted $\lim_{h\longrightarrow 0}(1+h)^{1/h}=e$. Calculate

$$\lim_{n\longrightarrow\infty} \left|\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right|^n. \tag{173}$$

Solution. First note that as $f(x)$ is continuous at x_0,

$$\lim_{n\longrightarrow\infty} \frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)} = 1 \tag{174}$$

which means there is $N\in\mathbb{N}$ such that for all $n>N$,

$$\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)} > 0. \tag{175}$$

Therefore

$$\lim_{n\longrightarrow\infty} \left|\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right|^n = \lim_{n\longrightarrow\infty} \left(\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right)^n. \tag{176}$$

Write

$$\begin{aligned} \left|\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right|^n &= \left|1+\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{f(x_0)}\right|^n \\ &= \left|1+\frac{1}{n}\,\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\,\frac{1}{f(x_0)}\right|^n. \end{aligned} \tag{177}$$

Now let

$$h_n = \frac{1}{n}\,\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\,\frac{1}{f(x_0)}. \tag{178}$$

We have

$$h_n \longrightarrow 0 \tag{179}$$

and

$$n = \frac{1}{h_n}\,\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\,\frac{1}{f(x_0)}. \tag{180}$$

Therefore

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\left|\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right|^n &= \lim_{n\longrightarrow\infty}\left(\frac{f\left(x_0+\frac{1}{n}\right)}{f(x_0)}\right)^n \\
&= \lim_{n\longrightarrow\infty}(1+h_n)^{\frac{1}{h_n}\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\frac{1}{f(x_0)}} \\
&= \lim_{n\longrightarrow\infty}\left[(1+h_n)^{1/h_n}\right]^{\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\frac{1}{f(x_0)}} \\
&= \left\{\lim_{h_n\longrightarrow 0}\left[(1+h_n)^{1/h_n}\right]\right\}^{\lim_{n\longrightarrow\infty}\left[\frac{f\left(x_0+\frac{1}{n}\right)-f(x_0)}{1/n}\frac{1}{f(x_0)}\right]} \\
&= \exp\left[f'(x_0)/f(x_0)\right]. \qquad (181)
\end{aligned}$$

Problem 42. (USTC) Let f be twice differentiable over $\mathbb{R}$, with $f(0)=f(1)=0$. Let $F(x)=x^2 f(x)$. Prove that there is $\xi\in(0,1)$ such that $F''(\xi)=0$.

Proof. All we need are x_1, x_2 such that $F'(x_1)=F'(x_2)=0$.

We calculate

$$F'(x)=2\,x\,f(x)+x^2\,f'(x). \qquad (182)$$

Thus it is clear that $F'(0)=0$.

On the other hand, $f(0)=f(1)=0$ implies $F(0)=F(1)=0$ which gives the existence of $\eta\in(0,1)$ such that $F'(\eta)=0$.

Now apply Mean Value Theorem again we obtain the existence of $\xi\in(0,\eta)\subset(0,1)$ satisfying

$$F''(\xi)=0. \qquad (183)$$

□

Remark 1. Note that the same idea can show the following: Let f be m-th differentiable with $f(0)=f(1)=0$, let $F(x)=x^m f(x)$, then there is $\xi\in(0,1)$ such that $F^{(m)}(\xi)=0$.

Problem 43. Let f be differentiable over $\mathbb{R}$. Then $f'(x)$, though may be not continuous, always satisfies the Intermediate Value Property:

For any s between $f'(a)$ and $f'(b)$, there is $\xi\in[a,b]$ such that $f'(\xi)=s$.

Then use this to prove: If f is differentiable in $(a,\,b)$ and $f'\neq 0$, then f is either increasing or decreasing.

Proof. Define the function

$$g(x)=\begin{cases} f(b)+f'(b)\,(x-b) & x>b \\ f(x) & x\in[a,b] \\ f(a)+f'(a)\,(x-a) & x<a \end{cases}. \qquad (184)$$

Then $g(x)$ is differentiable over $\mathbb{R}$. Now use Mean Value Theorem. The idea is very easy to understand if you draw the graph of the function $g(x)$.

If there are $x_1<x_2, x_3<x_4$ such that $f(x_1)<f(x_2)$, $f(x_3)>f(x_4)$, then by mean value theorem we have $\xi\in(x_1,\,x_2)$, $\eta\in(x_3,\,x_4)$ such that $f'(\xi)>0$, $f'(\eta)<0$. Now the mean value property implies the existence of x_0 between ξ,η such that $f'(x_0)=0$. Contradiction. □

Remark 2. A better way to prove is to consider $g(x)=f(x)-s\,x$ defined for $x\in[a,\,b]$. Assume $f'(a)<s<f'(b)$. Then we have $g'(a)<0, g'(b)>0$. Since g is continuous on $[a,b]$, there is a minimizer $\xi\in[a,b]$. All we need to show is $\xi\neq a,b$. Since $g'(a)<0$, for h small enough we have $g(a+h)<g(a)$ so $\xi\neq a$. Similarly $\xi\neq b$. Thus $g'(\xi)=0\Longrightarrow f'(\xi)=s$.

Problem 44. (USTC) Calculate

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right). \tag{185}$$

(Hint: Write $\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right)=\sum_{k=1}^{n}\frac{k\,a}{n^2}+\sum_{k=1}^{n}\left[\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right]$, try to estimate $\left|\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right|$ using Taylor polynomial)
Solution. Write

$$\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right)=\sum_{k=1}^{n}\frac{k\,a}{n^2}+\sum_{k=1}^{n}\left[\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right] \tag{186}$$

Now recall the Taylor expansion of $\sin x$ with Lagrange form of remainder (to degree 1):

$$\sin x = x-\frac{\sin\xi}{2}\,x^2 \tag{187}$$

for some $\xi\in(0,x)$. This gives

$$\left|\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right|\leqslant\frac{1}{2}\left(\frac{k\,a}{n^2}\right)^2\leqslant\frac{a^2}{2}\,\frac{1}{n^2}. \tag{188}$$

Now notice

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n}\frac{k\,a}{n^2}=\lim_{n\longrightarrow\infty}\frac{a}{n^2}\sum_{n=1}^{n}k=\lim_{n\longrightarrow\infty}\frac{a}{n^2}\,\frac{n\,(n+1)}{2}=\frac{a}{2}. \tag{189}$$

On the other hand

$$\left|\sum_{k=1}^{n}\left[\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right]\right|\leqslant\sum_{k=1}^{n}\frac{a^2}{2}\,\frac{1}{n^2}=\frac{a^2}{2\,n}. \tag{190}$$

Application of Squeeze Theorem gives

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n}\left[\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right]=0. \tag{191}$$

Therefore

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right)=\frac{a}{2}. \tag{192}$$

Problem 45. Let f be differentiable on $(0,\ \infty)$ with $\lim_{x\longrightarrow\infty}\ [f(x)\ +\ f'(x)]\ =\ 0$. Prove that $\lim_{x\longrightarrow\infty}f(x)=0$. (Hint: Let $F(x)=e^x\,f(x)$, $G(x)=e^x$. Apply Cauchy's generalized mean value theorem.)

Proof. Following the hint, we have for any $x>y>0$,

$$\frac{e^x\,f(x)-e^y\,f(y)}{e^x-e^y}=f(\xi)+f'(\xi)\qquad\text{for some }\xi\in(y,x). \tag{193}$$

Therefore for every ε, there is $M>0$ such that for all $x>y>M$,

$$\left|\frac{e^{x-y}\,f(x)-f(y)}{e^{x-y}-1}\right|=\left|\frac{e^x\,f(x)-e^y\,f(y)}{e^x-e^y}\right|<\varepsilon/2. \tag{194}$$

This gives

$$|f(x)|<\left|\frac{e^{x-y}}{e^{x-y}-1}\right|\,|f(x)|<\frac{\varepsilon}{2}+\frac{|f(y)|}{|e^{x-y}-1|} \tag{195}$$

Now fix $y=M+1$. Take $M'=M+1+\ln\left(\frac{2\,|f(y)|}{\varepsilon}+1\right)$, then for every $x>M'$, we have

$$\left|\frac{f(y)}{e^{x-y}-1}\right|<\varepsilon/2 \tag{196}$$

which leads to

$$|f(x)| < \varepsilon. \tag{197}$$

So by definition $\lim_{x \longrightarrow \infty} f(x) = 0$. □

Problem 46. Let f be continuous on $[0, \infty)$ and satisfy $\lim_{x \longrightarrow \infty} f(x) = a$. Prove

$$\lim_{x \longrightarrow \infty} \frac{1}{x} \int_0^x f(t)\, \mathrm{d}t = a. \tag{198}$$

Proof. For any $\varepsilon > 0$, since $\lim_{x \longrightarrow \infty} f(x) = a$, there is $M_1 > 0$ such that

$$|f(x) - a| < \varepsilon/2 \tag{199}$$

for all $x > M_1$. Because f is continuous on $[0, M_1]$, it is bounded on $[0, M_1]$, that is there is $A > 0$ such that

$$|f(x)| \leqslant A \tag{200}$$

for all $x \in [0, M]$.

Now take $M = \max\left\{M_1, \frac{2\, M_1\, (A + |a|)}{\varepsilon}\right\}$, we have for any $x > M$,

$$\begin{aligned} \left|\frac{1}{x} \int_0^x f(t)\, \mathrm{d}t - a\right| &= \left|\frac{1}{x} \int_0^x (f(t) - a)\, \mathrm{d}t\right| \\ &\leqslant \frac{1}{x} \int_0^x |f(t) - a|\, \mathrm{d}t \\ &= \frac{1}{x} \int_0^{M_1} |f(t) - a|\, \mathrm{d}t + \frac{1}{x} \int_{M_1}^x |f(t) - a|\, \mathrm{d}t \\ &\leqslant \frac{1}{x} \int_0^{M_1} (A + |a|)\, \mathrm{d}t + \frac{1}{x} \int_{M_1}^x \frac{\varepsilon}{2}\, \mathrm{d}t \\ &\leqslant \frac{M_1\, (A + |a|)}{x} + \frac{\varepsilon}{2} \\ &< \frac{M_1\, (A + |a|)}{M} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned} \tag{201}$$

□

Problem 47. (USTC) Let

$$F(x) = \int_0^x \frac{\sin t}{t}\, \mathrm{d}t, \qquad x \in (0, \infty). \tag{202}$$

Prove that $\max_{x \in \mathbb{R}} F = F(\pi)$.

Proof. First notice that

$$\frac{\sin t}{t} \begin{cases} \geqslant 0 & t \in [2\, k\, \pi, 2\, k\, \pi + \pi] \\ \leqslant 0 & t \in [2\, k\, \pi + \pi, 2\, (k+1)\, \pi] \end{cases}. \tag{203}$$

Therefore $F(x)$ is increasing in $[2\, k\, \pi, 2\, k\, \pi + \pi]$ and decreasing in $[2\, k\, \pi + \pi, 2\, (k+1)\, \pi]$ for every $k \in \mathbb{Z}$. All we need to show now is $F(\pi) > F(2\, k\, \pi + \pi)$ for every k. In fact we will show

$$F(\pi) > F(3\, \pi) > F(5\, \pi) > \cdots \tag{204}$$

We show $F(\pi) > F(3\, \pi)$ here, others can be done similarly. We have

$$\begin{aligned} F(3\, \pi) &= F(\pi) + \int_\pi^{2\pi} \frac{\sin t}{t}\, \mathrm{d}t + \int_{2\pi}^{3\pi} \frac{\sin t}{t}\, \mathrm{d}t \\ &= F(\pi) + \int_\pi^{2\pi} \frac{\sin t}{t}\, \mathrm{d}t + \int_\pi^{2\pi} \frac{\sin\,(x + \pi)}{x + \pi}\, \mathrm{d}x \\ &= F(\pi) + \int_\pi^{2\pi} \frac{\sin t}{t} - \frac{\sin t}{t + \pi}\, \mathrm{d}t < F(\pi). \end{aligned} \tag{205}$$

The last inequality follows from the fact that $\sin t < 0$ in $(\pi, 2\pi)$. □

Problem 48. $a_n \geqslant 0$, $\sum_{n=1}^{\infty} a_n$ converges. Let $b_n = \frac{a_n}{\sum_n^{\infty} a_k}$. Prove that $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. We show that b_n is not Cauchy through showing: For any $n \in \mathbb{N}$, there is $m > n$ such that

$$\sum_{k=n}^{m} b_k > \frac{1}{2}. \tag{206}$$

Take any $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{k=n}^{\infty} a_k$. Denote $S = \sum_{k=n}^{\infty} a_k > 0$. Now denote $S_m = \sum_{k=n}^{m} a_k$, we have S_m increasing and $S_m \longrightarrow S$. Thus there is $m \in \mathbb{N}$ such that $S_m > S/2$.

For this m, we have

$$\sum_{k=n}^{m} b_k = \sum_{k=n}^{m} \frac{a_k}{\sum_{l=k}^{\infty} a_l} \geqslant \sum_{k=n}^{m} \frac{a_k}{\sum_{l=n}^{\infty} a_l} = \frac{\sum_{k=n}^{m} a_k}{\sum_{k=n}^{\infty} a_k} = \frac{S_m}{S} > \frac{1}{2}. \tag{207}$$

Thus ends the proof. □

Problem 49. (Alternating series) Let $b_n \geqslant 0$ with $\lim_{n \longrightarrow \infty} b_n = 0$. Assume there is $N \in \mathbb{N}$ such that for all $n > N$, $b_n \geqslant b_{n+1}$.

a) Prove that $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

b) Apply this criterion to prove the convergence of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!}$.

c) Show that the condition "b_n is decreasing" cannot be dropped.

Proof.

a) For any $n = 2k > N$, we have

$$S_{2(k+1)} = \sum_{n=1}^{2(k+1)} (-1)^{n+1} b_n = S_{2k} + b_{2k+1} - b_{2k+2} \geqslant S_{2k}. \tag{208}$$

Thus S_{2k} is increasing when $2k > N$.

Now let k_0 be such that $2k_0 > N$ but $2(k_0 - 1) < N$. Now we have, for any $k \geqslant k_0$,

$$S_{2k} = S_{2k_0-1} - b_{2k_0} + b_{2k_0+1} - \cdots - b_{2k-2} + b_{2k-1} - b_{2k} \leqslant S_{2k_0-1} \tag{209}$$

which means S_{2k} is bounded above.

Therefore $S_{2k} \longrightarrow s \in \mathbb{R}$. Since $S_{2k+1} - S_{2k} = b_{2k+1} \longrightarrow 0$, we have $S_{2k+1} \longrightarrow s$ too. Combine these two we have $S_k \longrightarrow s$.

b) All we need to show is $\frac{1}{n}$ is decreasing with limit 0, which is obvious, and $\frac{3^n}{n!}$ is decreasing with limit 0. For the latter, notice that

$$\frac{3^n/n!}{3^{n+1}/(n+1)!} = \frac{n+1}{3} \tag{210}$$

which $\geqslant 1$ when $n \geqslant 2$.

On the other hand, for $n \geqslant 5$, we have

$$b_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3}{n+1} b_n \leqslant \frac{1}{2} b_n \tag{211}$$

which means $\lim_{n \longrightarrow \infty} b_n = 0$.

c) That the condition "b_n is decreasing" is necessary can be seen from the following example: $b_n = \begin{cases} 1/n & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$. □

Really Advanced

Problem 50. Let f be defined on (a,b) and $x_0 \in (a,b)$. Assume that $f^{(n+1)}(x)$ exists and is continuous on (a,b) with $f^{(n+1)}(x_0) \neq 0$. Consider the Taylor polynomial with Lagrange remainder:

$$f(x) = \cdots + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n. \tag{212}$$

Recall that ξ can be viewed as a function of x. If we define (naturally) $\xi(x_0) = x_0$, prove that $\xi(x)$ is differentiable at x_0 with

$$\xi'(x_0) = \frac{1}{n+1}. \tag{213}$$

Proof. We only need to show

$$\lim_{x \longrightarrow x_0} \frac{\xi(x) - x_0}{x - x_0} = \frac{1}{n+1}. \tag{214}$$

expand to degree n with Lagrange form of remainder

$$f(x) = \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x - x_0)^{n+1}. \tag{215}$$

Thus we have

$$\frac{f^{(n)}(\xi) - f^{(n)}(x_0)}{x - x_0} = \frac{f^{(n+1)}(\eta)}{n+1}. \tag{216}$$

On the other hand, by Mean Value Theorem there is $\eta' \in (x_0, \xi)$ such that

$$\frac{f^{(n)}(\xi) - f^{(n)}(x_0)}{\xi - x_0} = f^{(n+1)}(\eta'). \tag{217}$$

Therefore

$$\frac{\xi(x) - x_0}{x - x_0} = \frac{f^{(n+1)}(\eta)}{f^{(n+1)}(\eta')} \frac{1}{n+1}. \tag{218}$$

Note that η is a function of x, while η' is a function of ξ which is itself a function of x, so η' is also a function of x. We further have

$$x \longrightarrow x_0 \Longrightarrow \eta, \eta' \longrightarrow x_0 \tag{219}$$

by Squeeze Theorem.

Since $f^{(n+1)}$ is continuous with $f^{(n+1)}(x_0) \neq 0$, we have

$$\lim_{x \longrightarrow x_0} \frac{\xi(x) - x_0}{x - x_0} = \frac{\lim_{\eta \longrightarrow 0} f^{(n+1)}(\eta)}{\lim_{\eta' \longrightarrow 0} f^{(n+1)}(\eta')} \frac{1}{n+1} = \frac{f^{(n+1)}(x_0)}{f^{(n+1)}(x_0)} \frac{1}{n+1} = \frac{1}{n+1}. \tag{220}$$

So by definition $\xi(x)$ is differentiable with $\xi'(x_0) = \frac{1}{n+1}$. □

Problem 51. (USTC) Let f be differentiable. $a\,b > 0$. Then there is $\xi \in (a,b)$ such that

$$\frac{1}{a-b}[a\,f(b) - b\,f(a)] = f(\xi) - \xi\,f'(\xi). \tag{221}$$

(Hint: Use Cauchy's Generalized Mean Value Theorem).

Proof. Notice that

$$\frac{a\,f(b) - b\,f(a)}{a-b} = \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}}. \tag{222}$$

Now apply Cauchy's Generalized Mean Value Theorem to $f(x)/x$ and $1/x$, we have

$$\frac{\frac{f(b)}{b}-\frac{f(a)}{a}}{\frac{1}{b}-\frac{1}{a}}=\frac{\left(\frac{f(x)}{x}\right)'_{x=\xi}}{\left(\frac{1}{x}\right)'_{x=\xi}}=\frac{[f'(\xi)\,\xi-f(\xi)]/\xi^2}{-1/\xi^2}=f(\xi)-\xi\,f'(\xi). \tag{223}$$

□

Remark 3. Note that the condition $a\,b>0$ is necessary because $1/x$ is not differentiable on (a,b) if $a\,b<0$.

Problem 52. (USTC) Let $f(x)$ be differentiable on $[0,1]$. $f(0)=0, f(1)=1$. Then for any $n\in\mathbb{N}$ and $k_1,\ldots,k_n>0$, there are n distinct numbers $x_1,\ldots,x_n\in(0,1)$, such that

$$\sum_{i=1}^n \frac{k_i}{f'(x_i)}=\sum_{i=1}^n k_i. \tag{224}$$

Remark 4. Note that when $k=1$, this is simply mean value theorem. Also if we do not require $x_1,\ldots,$ x_n to be distinct, the problem is trivial since we can take $x_1=\cdots=x_n=\xi$ with $f'(\xi)=1$.

(Hint: Take $y_1<y_2<\cdots<y_{n-1}$ such that $f(y_i)=\frac{k_1+\cdots+k_i}{k_1+\cdots+k_n}$. Set $y_0=0, y_1=1$. Then define $g(x)$ to be linear on each $[y_i,y_{i+1}]$ with $g(y_i)=f(y_i), g(y_{i+1})=f(y_{i+1})$. Apply Cauchy's generalized mean value theorem.)

Proof. Following the hint, on each $[y_{i-1},y_i]$ we have a x_i such that

$$\frac{g'(x_i)}{f'(x_i)}=\frac{g(y_i)-g(y_{i-1})}{f(y_i)-f(y_{i-1})}=1. \tag{225}$$

But since g is linear, we have

$$g'(x_i)=\frac{k_i/\left(\sum_{j=1}^n k_j\right)}{y_i-y_{i-1}} \tag{226}$$

which gives

$$\frac{k_i}{f'(x_i)}=(y_i-y_{i-1})\left(\sum_{j=1}^n k_j\right)\Longrightarrow\sum_{i=1}^n\frac{k_i}{f'(x_i)}=\left(\sum_{i=1}^n (y_i-y_{i-1})\right)\left(\sum_{j=1}^n k_j\right)=\sum_{j=1}^n k_j \tag{227}$$

since $\sum_{i=1}^n (y_i-y_{i-1})=y_n-y_0=1-0=1.$ □

Problem 53. (USTC) Let f,g be continuous on $[-1,1]$, infinitely differentiable on $(-1,1)$, and

$$\left|f^{(n)}(x)-g^{(n)}(x)\right|\leqslant n!\,|x| \qquad n=0,1,2,\ldots \tag{228}$$

Prove that $f=g$. (Hint: Show first $f^{(n)}(0)=0$ for all n. Then use Taylor polynomial with Lagrange form of remainder)

Proof. Set $h(x)=f-g$. We have $h(0)=0$. Next we have

$$|h(x)-h(0)|=|h'(\xi)\,x|\leqslant x^2\Longrightarrow h'(0)=0. \tag{229}$$

Do this $n-1$ more times we have $h^{(n)}(0)=0$ for any $n\in\mathbb{N}$. Now estimate

$$|h(x)|=\left|\frac{h^{(n)}(\xi)}{n!}\,x^n\right|\leqslant|x|^{n+1}. \tag{230}$$

For every $|x|<1$, letting $n\longrightarrow\infty$, we have

$$|h(x)|\leqslant\lim_{n\longrightarrow\infty}|x|^n=0. \tag{231}$$

Therefore $h(x)=0$ for all $x\in(-1,1)$. As $h(x)$ is continuous on $[-1,1]$, we have $h(x)=0$ for all $x=-1$, 1 too. □

Problem 54. Define γ_n through $\sum_{k=1}^{n-1}\frac{1}{k}=\ln n+\gamma_n$

a) Show that $\gamma_n\geqslant 0$, γ_n is increasing with respect to n.

b) Show that $\gamma_n\longrightarrow\gamma\in\mathbb{R}$.

c) Show that $\sum_1^\infty(-1)^{n+1}/n=\ln 2$.

Proof.

a) Denote

$$a_n=\frac{1}{n}-\int_n^{n+1}\frac{1}{x}\,\mathrm{d}x. \tag{232}$$

Then we have $a_n\geqslant 0$ and

$$\gamma_n=\sum_{k=1}^{n-1}a_n. \tag{233}$$

Clearly γ_n is increasing.

b) All we need to show is γ_n is bounded above. We have

$$\gamma_n=\sum_1^{n-1}\frac{1}{k}-\int_1^n\frac{\mathrm{d}x}{x}\leqslant 1+\sum_2^n\frac{1}{k}-\int_1^n\frac{\mathrm{d}x}{x}=1+\sum_{k=2}^n\left[\frac{1}{k}-\int_{k-1}^k\frac{\mathrm{d}x}{x}\right]<1. \tag{234}$$

Therefore γ_n converges with some limit $\gamma\in(0,1)$.

c) We have

$$\sum_{k=1}^{2m}\frac{(-1)^{k+1}}{k}=\sum_{k=1}^{2m}\frac{1}{k}-\sum_{k=1}^{m}\frac{1}{k}=\ln(2m+1)-\ln(m+1)=\ln\left(\frac{2m+1}{m+1}\right); \tag{235}$$

$$\sum_{k=1}^{2m+1}\frac{(-1)^{k+1}}{k}=\sum_{k=1}^{2m}\frac{1}{k}-\sum_{k=1}^{m}\frac{1}{k}+\frac{1}{2m+1}=\ln\left(\frac{2m+1}{m+1}\right)+\frac{1}{2m+1}. \tag{236}$$

Since

$$\lim_{m\longrightarrow\infty}\ln\left(\frac{2m+1}{m+1}\right)=\ln 2, \tag{237}$$

for any $\varepsilon>0$, if we take $N>\max\left\{N_1,\frac{2}{\varepsilon}\right\}$ where N_1 is such that for any $m>N_1/2$,

$$\left|\ln\left(\frac{2m+1}{m+1}\right)-\ln 2\right|<\varepsilon/2 \tag{238}$$

Then for any $n > N$ we have

$$\left| \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - \ln 2 \right| < \varepsilon. \tag{239}$$

Thus ends the proof. □

Remark 5. The convergence of γ_n can be shown directly by estimating

$$a_n = \frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,\mathrm{d}x = \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \left[\ln\left(1 + \frac{1}{n}\right) - \ln 1\right] \tag{240}$$

using Mean Value Theorem (on $\ln$).

Problem 55. (Bonar2006)

a) Let $\sum_{n=1}^{\infty} a_n$ be any convergent non-negative series, then there is another convergent non-negative series $\sum_{n=1}^{\infty} A_n$ satisfying $\lim_{n \longrightarrow \infty} (A_n / a_n) = \infty$; (Hint: Set $A_n = \frac{a_n}{\sqrt{a_n + a_{n+1} + \cdots}}$)

b) Let $\sum_{n=1}^{\infty} D_n$ be any divergent non-negative series, then there is another divergent non-negative series $\sum_{n=1}^{\infty} d_n$ satisfying $\lim_{n \longrightarrow \infty} (d_n / D_n) = 0$. (Hint: Set $d_n = D_n / (D_1 + \cdots + D_{n-1})$)

Proof.

a) Define

$$t_n = \sum_{k=n}^{\infty} a_n \tag{241}$$

and then

$$A_n = \frac{a_n}{\sqrt{t_n}}. \tag{242}$$

Then clearly $\lim_{n \longrightarrow \infty} (A_n / a_n) = \infty$.

On the other hand, we have

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{t_k - t_{k-1}}{\sqrt{t_k}} \leqslant \int_0^{t_1} \frac{1}{\sqrt{x}}\,\mathrm{d}x < \infty. \tag{243}$$

b) Define

$$S_n = \sum_{k=1}^{n} D_k \tag{244}$$

and then

$$d_n = \frac{D_n}{S_{n-1}}. \tag{245}$$

Since $\sum_{n=1}^{\infty} D_n$ diverges, together with $D_n \geqslant 0$ we have $S_n \longrightarrow \infty$ so $\lim_{n \longrightarrow \infty} (d_n / D_n) = 0$.

On the other hand,

$$\sum_{k=1}^{\infty} d_n > \int_{S_1}^{S_n} \frac{1}{x}\,\mathrm{d}x \longrightarrow \infty \tag{246}$$

as $n \longrightarrow \infty$. □

Really Really Advanced

Problem 56. (USTC) Let $f(x)$ be continuous on $[0,\infty)$ and be bounded. Then for every $\lambda\in\mathbb{R}$, there is $x_n\longrightarrow\infty$ such that

$$\lim_{n\longrightarrow\infty}[f(x_n+\lambda)-f(x_n)]=0. \tag{247}$$

Proof. Prove by contradiction. Note that all we need to show is that the sets

$$A_n:=\{x\geqslant n, |f(x+\lambda)-f(x)|\leqslant 1/n\} \tag{248}$$

are non-empty for every $n\in\mathbb{N}$. Assume the contrary: There is $n_0\in\mathbb{N}$ such that $A_{n_0}=\varnothing$. That is for all $x>n_0$, either

$$f(x+\lambda)-f(x)>1/n_0 \quad \text{or} \quad f(x+\lambda)-f(x)<1/n_0. \tag{249}$$

Let $g(x):=f(x+\lambda)-f(x)$. Then $g(x)$ is continuous. We have for every $x>n_0$, either $g(x)>1/n_0$ or $g(x)<-1/n_0$. There are three possible cases:

- If there are $x_1>n_0$ and $x_2>n_0$ such that $g(x_1)>1/n_0$, $g(x_2)<-1/n_0$, then by intermediate value theorem we have $\xi>n_0$ such that $g(\xi)=0$. Contradiction.
- If $g(x)>1/n_0$ for all $x>n_0$, then we have

  $$f(n_0+k\,\lambda)>f(n_0+(k-1)\,\lambda)+\frac{1}{n_0}>\cdots>f(n_0)+\frac{k}{n_0}. \tag{250}$$

  As a consequence, for any $M\in\mathbb{R}$, take $k>(|M|+|f(n_0)|)\,n_0$, we have

  $$f(n_0+k\,\lambda)>M. \tag{251}$$

  This means f is not bounded above and thus not bounded. Contradiction.
- $g(x)<-1/n_0$ for all $x>n_0$. Similar. □

Problem 57. Let $f(x)$ be differentiable with $f(x_0)=0$. Further assume $|f'(x)|\leqslant|f(x)|$ for all $x>x_0$. Prove that $f(x)=0$ for all $x\geqslant x_0$.

Proof. We prove that $f(x)=0$ for all $x\in[x_0,x_0+1/2]$. Then by repeating the same argument setting with x_0 replaced by $x_0+1/2$ we will get $f(x)=0$ for all $x\in[x_0,x_0+1]$. Doing this again and again we can cover all $x\geqslant x_0$.

Take any $x\in(x_0,x_0+1/2]$. By mean value theorem we have

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(\xi) \tag{252}$$

for some $\xi_1\in(x_0,x_0+1/2)$. This leads to

$$|f(x)| \;=\; |f'(\xi)|\,|x-x_0|\leqslant\frac{1}{2}\,|f(\xi_1)|.$$

Apply the same argument we find $\xi_2\in(x_0,\xi_1)$ such that

$$|f(\xi_1)|\leqslant\frac{1}{2}\,|f(\xi_2)|. \tag{253}$$

This way we obtain a decreasing sequence ξ_n satisfying

$$|f(\xi_{n-1})|\leqslant\frac{1}{2}\,|f(\xi_n)|. \tag{254}$$

This implies

$$|f(x)|\leqslant\frac{1}{2^n}\,|f(\xi_n)|. \tag{255}$$

Since $f(x)$ is differentiable it is continuous on $[x_0,x_0+1/2]$ which means there is $A>0$ such that

$$|f(\xi_n)|\leqslant A \qquad \forall n\in\mathbb{N}. \tag{256}$$

This gives

$$|f(x)| \leqslant \frac{A}{2^n} \tag{257}$$

for all n, so $f(x) = 0$. □

Problem 58. (Darboux's Theorem) [1]Let $f(x)$ be a bounded function over a finite interval $[a, b]$. Let $P_n = \left\{x_0 = a, x_1 = a + \frac{b-a}{n}, \ldots, x_n = b\right\}$. Then

$$U(f, P_n) \longrightarrow U(f); \qquad L(f, P_n) \longrightarrow L(f). \tag{258}$$

Proof. Let $M > 0$ be the bound of $|f(x)|$. We prove the first statement, the second is similar.

Take any $\varepsilon > 0$. Let $P = \{x_1, \ldots, x_m\}$ be a partition such that

$$U(f, P) \leqslant U(f) + \varepsilon/2. \tag{259}$$

Now consider P_n with $n > 2m$. It is clear that at least $n - 2m$ intervals in P_n are fully contained in some $[x_{i-1}, x_i]$ of P. As a consequence

$$\begin{aligned} U(f, P_n) &= \sum_{\text{intervals contained in some } [x_{i-1}, x_i]} + \sum_{\text{intervals containing some } x_i} \\ &\leqslant U(f, P) + \frac{2mM}{n} \\ &\leqslant U(f) + \frac{2mM}{n} + \frac{\varepsilon}{2}. \end{aligned} \tag{260}$$

From this we see that, if we take $N \in \mathbb{N}$ such that $N > \frac{4mM}{\varepsilon}$, then for every $n > N$,

$$U(f, P_n) \leqslant U(f) + \varepsilon. \tag{261}$$

On the other hand, by definition of $U(f)$ we have

$$U(f, P_n) \geqslant U(f). \tag{262}$$

Thus we have shown that, for every $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n > N$,

$$|U(f, P_n) - U(f)| < \varepsilon. \tag{263}$$

This by definition gives $U(f, P_n) \longrightarrow U(f)$ as $n \longrightarrow \infty$. □

Problem 59. (Claesson1970) Let $f(x)$ be a bounded function over a finite interval $[a, b]$. Let $U(f)$ denote its upper integral. Prove: f is integrable $\Longleftrightarrow$ For any bounded function $g(x)$,

$$U(f + g) = U(f) + U(g). \tag{264}$$

Proof.

- $\Longrightarrow$. Assume f is integrable. Let P be any partition of $[a, b]$. We have by definition

  $$U(f + g, P) \leqslant U(f, P) + U(g, P). \tag{265}$$

  On the other hand, by definition

  $$\begin{aligned} U(f + g, P) &= \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x) + g(x)\right](x_i - x_{i-1}) \\ &\geqslant \sum_{i=1}^{n} \left[\inf_{x \in [x_{i-1}, x_i]} f(x) + \sup_{x \in [x_{i-1}, x_i]} g(x)\right](x_i - x_{i-1}) \\ &= L(f, P) + U(g, P). \end{aligned} \tag{266}$$

1. Darboux Theorem actually states that the conclusion holds for any sequence of partitions with $\sup_i (x_i - x_{i-1}) \longrightarrow 0$. But the proof in such general case is very similar to the special one here.

Thus we have

$$L(f, P) + U(g, P) \leqslant U(f + g, P) \leqslant U(f, P) + U(g, P) \tag{267}$$

for any partition P. Thus on one hand we have

$$U(f + g) \leqslant U(f + g, P) \leqslant U(f, P) + U(g, P) \tag{268}$$

which leads to

$$U(f + g) \leqslant U(f) + U(g) \tag{269}$$

due to the arbitrariness of P.

On the other hand, we have

$$U(f + g, P) \geqslant L(f, P) + U(g, P) \geqslant L(f, P) + U(g). \tag{270}$$

Now for any partition Q, we have

$$U(f + g, Q) \geqslant U(f + g, P \cup Q) \geqslant L(f, P \cup Q) + U(g) \geqslant L(f, P) + U(g). \tag{271}$$

Taking supreme over P and then infimum over Q we obtain

$$U(f + g) \geqslant L(f) + U(g). \tag{272}$$

Summarizing:

$$L(f) + U(g) \leqslant U(f + g) \leqslant U(f) + U(g). \tag{273}$$

But f is integrable so $L(f) = U(f)$ which leads to

$$U(f + g) = U(f) + U(g). \tag{274}$$

- $\Longleftarrow$. Take $g = -f$. Then g is bounded. We have

$$0 = U(0) = U(f + g) = U(f) + U(-f). \tag{275}$$

In the following we show $U(-f) = -L(f)$ with when substituted into (275) immediately gives integrability of f.

Now notice, for any partition P,

$$\begin{aligned} U(-f, P) &= \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} (-f(x)) \right] (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \left[-\inf_{x \in \{x_{i-1}, x_i\}} f(x) \right] (x_i - x_{i-1}) \\ &= -L(f, P). \end{aligned} \tag{276}$$

From this we have

$$U(-f) \leqslant -L(f, P) \tag{277}$$

for any partition P so

$$U(-f) \leqslant -L(f) \Longrightarrow L(f) \leqslant -U(-f). \tag{278}$$

On the other hand,

$$L(f, P) = -U(-f, P) \Longrightarrow L(f) \geqslant -U(-f, P) \text{ for all } P \Longrightarrow L(f) \geqslant -U(-f). \tag{279}$$

Summarizing, we have

$$U(-f) = -L(f). \tag{280}$$

Thus we have

$$0 = U(f) - L(f) \Longrightarrow U(f) = L(f) \tag{281}$$

so f is integrable. □