Math 314 Fall 2012 Midterm Solutions

Oct. 23, 2012 2pm - 3:20pm. Total 60 Pts

NAME: ID#:

- Please write clearly and show enough work.

Problem 1. (5 pts) A function $f(x): E \mapsto \mathbb{R}$ is said to be Lipschitz continuous if there is $M \in \mathbb{R}$ such that for every $x, y \in E$, $|f(x) - f(y)| \leqslant M\ |x - y|$. Write down the logical statement of "$f(x)$ is not Lipschitz continuous".

Solution.
Lipschitz continuous: $\exists M \in \mathbb{R}\ \forall x, y \in E, \qquad |f(x) - f(y)| \leqslant M\ |x - y|$;
Not Lipschitz continuous: $\forall M \in \mathbb{R}\ \exists x, y \in E, \qquad |f(x) - f(y)| > M\ |x - y|$

Problem 2. (5 pts) Let $f(x): X \mapsto Y$ satisfy: For any $A, B \subseteq X$, if $A \cap B = \varnothing$ then $f(A) \cap f(B) = \varnothing$. Prove that f is one-to-one.

Proof. Assume the contrary, that is there are $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Take $A = \{x_1\}, B = \{x_2\}$, then $A \cap B = \varnothing$ but $f(A) \cap f(B) = \{f(x_1)\} \neq \varnothing$. Contradiction. ☐

Remark 1. It is equally simple to prove directly. Take $A = \{x_1\}$, $B = \{x_2\}$. $x_1 \neq x_2 \Longrightarrow A \cap B = \varnothing \Longrightarrow f(A) \cap f(B) = \varnothing \Longrightarrow f(x_1) \neq f(x_2)$.

Remark 2. It is not correct to say: $x_1 \in A, x_2 \in B$, because $A \cap B = \varnothing$, ... The reason is you need to show the existence of such A, B (satisfying $x_1 \in A, x_2 \in B$, because $A \cap B = \varnothing$).

Problem 3. (10 pts) Find the following limits. Justify your answers. (You can use the convergence/divergence of $x_n = n^a$ without proof)

a) **(3 pts)** $\lim_{n \longrightarrow \infty} \left[\sqrt{n^2 + 4n} - \sqrt{n^2 - 2n}\right]$.
Solution. We have

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \left[\sqrt{n^2 + 4n} - \sqrt{n^2 - 2n}\right] &= \lim_{n \longrightarrow \infty} \frac{\left[\sqrt{n^2 + 4n} - \sqrt{n^2 - 2n}\right]\left[\sqrt{n^2 + 4n} + \sqrt{n^2 - 2n}\right]}{\left[\sqrt{n^2 + 4n} + \sqrt{n^2 - 2n}\right]} \\
&= \lim_{n \longrightarrow \infty} \frac{(n^2 + 4n) - (n^2 - 2n)}{\sqrt{n^2 + 4n} + \sqrt{n^2 - 2n}} \\
&= \lim_{n \longrightarrow \infty} \frac{6n}{\sqrt{n^2 + 4n} + \sqrt{n^2 - 2n}} \\
&= \lim_{n \longrightarrow \infty} \frac{6}{\sqrt{1 + 4/n} + \sqrt{1 - 2/n}} \\
&= \frac{\lim_{n \to \infty} 6}{\lim_{n \to \infty}\left[\sqrt{1 + 4/n} + \sqrt{1 - 2/n}\right]} \\
&= \frac{6}{2} = 3. \qquad (1)
\end{aligned}$$

b) **(3 pts)** $\lim_{x \longrightarrow 1} \frac{x^2-1}{x^3-1}$.

Solution. We have

$$\begin{aligned} \lim_{x \longrightarrow 1} \frac{x^2-1}{x^3-1} &= \lim_{x \longrightarrow 1} \frac{(x+1)(x-1)}{(x^2+x+1)(x-1)} \\ &= \lim_{x \longrightarrow 1} \frac{x+1}{x^2+x+1} \\ &= \frac{\lim_{x \longrightarrow 1}(x+1)}{\lim_{x \longrightarrow 1}(x^2+x+1)} \\ &= \frac{2}{3}. \end{aligned} \tag{2}$$

c) **(4 pts)** $\lim_{n \longrightarrow \infty} \frac{5\,x_n}{n^3+2\,n+1}$ where x_n satisfies $|x_n| \leqslant 3\,n$ for all $n \in \mathbb{N}$.

Solution. Since $|x_n| \leqslant 3\,n$, we have $-15\,n \leqslant 5\,x_n \leqslant 15\,n$. Therefore

$$\frac{-15\,n}{n^3+2\,n+1} \leqslant \frac{5\,x_n}{n^3+2\,n+1} \leqslant \frac{15\,n}{n^3+2\,n+1} \tag{3}$$

This simplifies to

$$\frac{-15/n^2}{1+2/n^2+1/n^3} \leqslant \frac{5\,x_n}{n^3+2\,n+1} \leqslant \frac{15/n^2}{1+2/n^2+1/n^3}. \tag{4}$$

Taking limit $n \longrightarrow \infty$, by Squeeze Theorem we obtain

$$\frac{5\,x_n}{n^3+2\,n+1} \longrightarrow 0. \tag{5}$$

Problem 4. (10 pts) Let $A = \{x \in \mathbb{R}: e^{x^2} > e\}$, $B = \{x \in \mathbb{R}: x > 0, \ln x \leqslant 0\}$.

a) **(4 pts)** Express $A, B, A \cap B, A \cup B$ using intervals.

b) **(6 pts)** Among the four sets above, which is/are open? Which is/are closed? Justify your answers.

Solution.

a) $A = (-\infty, -1) \cup (1, \infty)$; $B = (0, 1]$; $A \cap B = \varnothing$; $A \cup B = (-\infty, -1) \cup (0, \infty)$.

b) $A, A \cup B$ are open since they are unions of open intervals. $A \cap B$ is both open and closed by definition. B is neither.

B is not open: Take $x_0 = 1 \in B$. Then for any (a, b) containing x_0, $b > x_0 = 1$. Thus $(a, b) \ni \frac{b+1}{2} > 1$ which means $\frac{b+1}{2} \notin B$. That is $(a, b) \not\subseteq B$.

B is not closed: We have $B^c = (-\infty, 0] \cup (1, \infty)$. We have $0 \in B^c$. For any (a, b) containing 0, $a < 0$ so $(a, b) \ni \frac{a}{2} \notin B^c$. So B^c is not open therefore B is not closed.

Remark 3. Since in my notes there is a lemma saying "half-open half-closed intervals are neither open nor closed", it's OK to simply say "B is neither because it is half-open half-closed."

Problem 5. (10 pts) Let $x_n = (-1)^n - e^{-n}$ and $E = \{x_n : n \in \mathbb{N}\}$. ($\mathbb{N} = \{1, 2, 3, \ldots\}$)

a) **(6 pts)** Find $\max E, \sup E, \min E, \inf E$. Justify your answers.

b) **(4 pts)** Calculate $\limsup_{n \longrightarrow \infty} x_n$ and $\liminf_{n \longrightarrow \infty} x_n$.

Solution.

a)

- $\max E$ does not exist. Assume the contrary, that is $x_{n_0} = \max E$. Then we have $x_{n_0+2} > x_{n_0}$. Contradiction.
- $\sup E = 1$. Since $1 \geqslant (-1)^n \geqslant (-1)^n - e^{-n}$ for all $n \in \mathbb{N}$, 1 is a upper bound. Now for any upper bound b, we have

$$b \geqslant (-1)^{2k} - e^{-2k} = 1 - e^{-2k}. \tag{6}$$

  for all $k \in \mathbb{N}$. Taking limit $k \longrightarrow \infty$, by comparison theorem we have $b \geqslant 1$.
- $\min E = x_1 = -1 - e^{-1}$. We have

$$x_1 = -1 - e^{-1} \leqslant (-1)^n - e^{-n} \tag{7}$$

  for all $n \in \mathbb{N}$ since $-1 \leqslant (-1)^n, e^{-1} \geqslant e^{-n} \Longrightarrow -e^{-1} \leqslant -e^{-n}$.
- Since $\min E$ exists, $\inf E = \min E = -1 - e^{-1}$.

b) We have

$$\sup \{x_n, x_{n+1}, \ldots\} \geqslant x_{2n} = 1 - e^{-2n}. \tag{8}$$

On the other hand $\sup \{x_n, \ldots\} \leqslant \sup E = 1$. Therefore comparison theorem gives

$$\lim_{n \longrightarrow \infty} (1 - e^{-2n}) \leqslant \limsup_{n \longrightarrow \infty} x_n \leqslant \lim_{n \longrightarrow \infty} 1 \Longrightarrow \limsup_{n \longrightarrow \infty} x_n = 1. \tag{9}$$

We have

$$\inf \{x_n, x_{n+1}, \ldots\} \leqslant x_{2n+1} = -1 - e^{-2n-1}. \tag{10}$$

On the other hand for any $k \geqslant n$, we have

$$x_k = (-1)^k - e^{-k} \geqslant -1 - e^{-k} \geqslant -1 - e^{-n}. \tag{11}$$

Thus comparison theorem gives

$$\lim_{n \longrightarrow \infty} (-1 - e^{-2n-1}) \leqslant \liminf_{n \longrightarrow \infty} x_n \leqslant \lim_{n \longrightarrow \infty} (-1 - e^{-n}) \Longrightarrow \liminf_{n \longrightarrow \infty} x_n = -1. \tag{12}$$

Problem 6. (10 pts) Let $x_0 = 25$ and define x_n through

$$x_{n+1} = \frac{3\,x_n}{7} - 8. \tag{13}$$

Prove that $\{x_n\}$ converges and find its limit. (You can use the formula $1 + r + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$ without proof)

Proof. We have

$$|x_{n+1} - x_n| = \frac{3}{7} |x_n - x_{n-1}| = \cdots = \left(\frac{3}{7}\right)^n |x_1 - x_0|. \tag{14}$$

For any $\varepsilon > 0$, take $N \geqslant \log_{(7/3)} \left[\frac{7\,|x_1 - x_0|}{4\,\varepsilon} \right]$. Then for any $m > n > N$, we have

$$\begin{aligned} |x_m - x_n| &= \left[\left(\frac{3}{7} \right)^{m-1} + \cdots + \left(\frac{3}{7} \right)^{n} \right] |x_1 - x_0| \\ &= \left(\frac{3}{7} \right)^{n} \left[1 + \frac{3}{7} + \cdots + \left(\frac{3}{7} \right)^{m-n-1} \right] |x_1 - x_0| \\ &= \left(\frac{3}{7} \right)^{n} \frac{1 - (3/7)^{m-n}}{1 - 3/7} |x_1 - x_0| \\ &\leqslant \left(\frac{3}{7} \right)^{n} \frac{7}{4} |x_1 - x_0| < \varepsilon. \end{aligned} \tag{15}$$

Therefore $\{x_n\}$ is Cauchy and consequently converge to some $a \in \mathbb{R}$.

Taking limit of both sides of $x_{n+1} = \frac{3\,x_n}{7} - 8$ we have $a = \frac{3\,a}{7} - 8 \Longrightarrow a = -14$. □

Remark 4. There are many alternative methods. To list a few:

- Guess $a = -14$. Then we have $x_{n+1} + 14 = \frac{3}{7}\,(x_n + 14)$. Can prove directly $x_n \longrightarrow -14$.
- Show x_n decreasing by math induction. Show $x_n + 14 \geqslant 0$ for all n that is -14 is a lower bound. Then x_n converges.
- Write

$$x_{n+1} = \frac{3\,x_n}{7} - 8 = \left(\frac{3}{7} \right)^2 x_{n-1} - \left[\frac{3}{7} + 1 \right] \cdot 8 = \cdots = \left(\frac{3}{7} \right)^{n+1} x_0 - \left[\left(\frac{3}{7} \right)^{n} + \cdots + 1 \right] \cdot 8 \tag{16}$$

then take limit directly.

Problem 7. (5 pts) Is $f(x) = \begin{cases} \dfrac{(\cos x)(\sin x^2)}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$ continuous for all $x \in \mathbb{R}$? Justify your answer. (You can use $\lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1$ without proof).

Solution. Yes.

Since $\sin x, x^2$ are continuous everywhere, the composite function $\sin x^2$ is continuous everywhere. Together with the continuity of $\cos x$ and x^2, we see that $f(x)$ is continuous at every $x \neq 0$.

At $x = 0$, we have $\lim_{x \longrightarrow 0} \cos x = 1$. So all we need to show is $\lim_{x \longrightarrow 0} \frac{\sin x^2}{x^2} = 1$. Let $g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ then $g(x)$ is continuous at 1. Therefore so does the composite function $g(x^2)$ which means $\lim_{x \longrightarrow 0} \frac{\sin x^2}{x^2} = 1$.

Remark 5. No point is deducted for the misunderstanding of $\sin x^2$ as $(\sin x)^2$.

Problem 8. (5 pts) Let $f: \mathbb{R} \mapsto \mathbb{R}$, $g: \mathbb{R} \mapsto \mathbb{R}$ be continuous functions. Assume $f(x) > 0$ for all $x \in \mathbb{R}$.

a) **(4 pts)** Prove that for any closed interval $[a, b]$ with $a, b \in \mathbb{R}$, there is $\delta_0 > 0$ such that for all $0 \leqslant \delta < \delta_0$, $f(x) + \delta\, g(x) > 0$ for all $x \in [a, b]$.

b) **(1 pt)** Is the claim still true when $a = -\infty$ or $b = \infty$ (or both)?

Proof.

a) For any closed interval $[a, b]$, we have

$$\min_{x \in [a,b]} f(x) = f(x_1), \qquad \min_{x \in [a,b]} g(x) = g(x_2) \tag{17}$$

for some $x_1, x_2 \in [a, b]$ due to the fact that f, g are continuous. Since $f > 0$ we have $f(x_1) > 0$. Take

$$\delta_0 = \begin{cases} -\dfrac{f(x_1)}{g(x_2)} & g(x_2) < 0 \\ 1 & g(x_2) \geqslant 0 \end{cases} \tag{18}$$

then for any $0 \leqslant \delta < \delta_0$,

$$f(x) + \delta\, g(x) > \begin{cases} f(x) + \delta_0\, g(x) \geqslant \min f(x) + \delta_0 \min g(x) = f(x_1) - \dfrac{f(x_1)}{g(x_2)}\, g(x_2) = 0 & \min g < 0 \\ f(x) \geqslant \min f(x) > 0 & \min g \geqslant 0 \end{cases}. \tag{19}$$

b) Not true anymore. Take $f(x) = e^{-x^2}$ and $g(x) = 1$. □