Math 314 A1 Fall 2012 Homework 7

Due Friday Nov. 30.

5:30pm (Assignment box CAB 3rd floor)

- Related sections in notes: §5.3 – §6.2.

Problem 1. (5 pts) Let $r<0$. Consider the function $f(x)=x^r:(0,\infty)\mapsto(0,\infty)$.

a) **(2 pts)** For what values of r is $f(x)$ improperly integrable on $(0,1)$? Justify your answer.

b) **(2 pts)** For what values of r is $f(x)$ improperly integrable on $(1,\infty)$? Justify your answer.

c) **(1 pt)** For what values of r is $f(x)$ improperly integrable on $(0,\infty)$? Justify your answer.

Problem 2. (3 pts) Let $g(x)$ be improperly integrable on $[0,\infty)$ and $f(x)$ be integrable on $[0,b]$ for every $0<b$. Furthermore $0\leqslant f(x)\leqslant g(x)$.

a) **(2 pts)** Prove that $f(x)$ is improperly integrable on $[0,\infty)$ and

$$\int_0^\infty f(x)\,\mathrm{d}x\leqslant\int_0^\infty g(x)\,\mathrm{d}x. \tag{1}$$

(Hint: You may want to use the "Cauchy" statement for limits of functions in midterm review)

b) **(1 pt)** Does the conclusion still hold if we drop $0\leqslant$? Justify your answer.

Problem 3. (3 pts) Let $\sum_{n=1}^\infty a_n$ be a infinite series. Let $\sum_{n=1}^\infty b_n$ be the series obtained from $\sum_{n=1}^\infty a_n$ by dropping all the zero terms (For example if $a_n=\frac{(-1)^n+1}{n}$, then $b_n=\frac{1}{n}$). Prove that $\sum_{n=1}^\infty a_n$ converges $\Longleftrightarrow \sum_{n=1}^\infty b_n$ converges.

Problem 4. (3 pts) Let $\sum_{n=1}^\infty a_n$, $\sum_{n=1}^\infty b_n$ be non-negative series with $a_n>0, b_n>0$ for all $n\in\mathbb{N}$.

a) **(2 pts)** If there is $N_0\in\mathbb{N}$ such that for all $n\geqslant N_0$, $\frac{a_{n+1}}{a_n}\leqslant\frac{b_{n+1}}{b_n}$, then $\sum_{n=1}^\infty b_n$ converges $\Longrightarrow \sum_{n=1}^\infty a_n$ converges;

b) **(1 pt)** Use a) to prove convergence for $\sum_{n=1}^\infty a_n$ with $a_1=1$ and

$$a_n=\frac{1}{4}\frac{2}{5}\cdots\frac{n-1}{n+2} \tag{2}$$

(Hint: use $b_n=\frac{1}{n\,(n+1)}$.)

Problem 5. (3 pts)

a) **(1 pt)** Show that $\sum_{n=1}^\infty (a_n\,b_n)$ converges if $A_n:=\sum_{k=1}^n a_k$ is bounded (that is there is $M\in\mathbb{R}$ such that $|A_n|\leqslant M$ for all $n\in\mathbb{N}$) and $b_n\geqslant 0$ is decreasing with $\lim_{n\longrightarrow\infty}b_n=0$.

b) **(2 pts)** Study the convergence of

$$\sum_{n=1}^\infty\frac{\cos(n\,\alpha)}{n} \tag{3}$$

for $\alpha\in[0,\pi)$.

Problem 6. (3 pts) Prove the following "integral test": For an infinite series $\sum_{n=1}^\infty a_n$ with $a_n\geqslant 0$, if there is a decreasing function $f(x)\geqslant 0$ such that $a_n=f(n)$, then $\sum_{n=1}^\infty a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,\mathrm{d}x$ exists and is finite. Furthermore

$$\int_1^\infty f(x)\,\mathrm{d}x<\sum_{n=1}^\infty a_n<\int_1^\infty f(x)\,\mathrm{d}x+a_1 \tag{4}$$