5. Integration

5.1. Riemann Integral.

5.1.1. Definitions.

Definition 5.1. (Partition) *Let $a, b \in \mathbb{R}$ with $a < b$. A partition of the interval $[a, b]$ is the set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_n = b. \tag{5.1}$$

A refinement of a partition P is a partition $Q \supseteq P$. In this case we say Q is finer than P.

The most useful property of partitions is the following.

Lemma 5.2. *Let P, Q be partitions. Then $P \cup Q$ is finer than both P and Q.*

Proof. This is obvious since partitions are defined as sets. □

Definition 5.3. (upper/lower Riemann sum) *Let $a, b \in \mathbb{R}$ with $a < b$, let $P = \{x_0, \ldots, x_n\}$ be a partition. Suppose $f: [a, b] \mapsto \mathbb{R}$ is bounded.*

i. The upper Riemann sum of f over P is the number

$$U(f, P) := \sum_{j=1}^{n} M_j(f)\,(x_j - x_{j-1}) \tag{5.2}$$

where

$$M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) \tag{5.3}$$

ii. The lower Riemann sum of f over P is the number

$$L(f, P) := \sum_{j=1}^{n} m_j(f)\,(x_j - x_{j-1}) \tag{5.4}$$

where

$$m_j(f) = \inf_{x \in [x_{j-1}, x_j]} f(x). \tag{5.5}$$

Remark 5.4. Note that since we assume f to be bounded, $m_j(f)$ and $M_j(f)$ are real numbers.

Since $m_j(f) \leqslant M_j(f)$, we clearly have

Lemma 5.5. $U(f, P) \geqslant L(f, P)$.

Definition 5.6. (Upper and lower integral) *We define the upper integral $U(f)$ of f over $[a, b]$ by*

$$U(f) := \inf \{U(f, P): P \text{ is a partition of } [a, b]\}. \tag{5.6}$$

We define the lower integral $L(f)$ of f over $[a, b]$ by

$$L(f) := \sup \{L(f, P): P \text{ is a partition of } [a, b]\}. \tag{5.7}$$

Theorem 5.7. *Upper and lower integrals exist for every bounded function f.*

Proof. All we need to show is that $U(f, P)$ is bounded from below and $L(f, P)$ is bounded from above. Take any bounded function f. There is $M > 0$ such that $|f| \leqslant M$, that is $-M \leqslant f \leqslant M$ for all $x \in [a, b]$.

Now by definition

$$U(f, P) = \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x)\right] (x_l - x_{l-1}) \geqslant -M \sum_{i=1}^{n} (x_l - x_{l-1}) = -M\,(b - a). \tag{5.8}$$

Similarly we can prove $L(f, P) \leqslant M\,(b-a)$. □

Theorem 5.8. $U(f) \geqslant L(f)$.

Naïvely, we only need to take inf and sup of left/right side of $U(f, P) \geqslant L(f, P)$. However this is wrong. If we imagine the process of taking inf as finding P_n such that $U(f, P_n) \longrightarrow U(f)$, then the fact that it's the same P on both sides leads to

$$U(f) = \lim_{n \longrightarrow \infty} U(f, P_n) \geqslant \lim_{n \longrightarrow \infty} L(f, P_n) \tag{5.9}$$

which, even if it exists, is usually not $L(f) = \inf_P L(f,\ P)$. A related issue is, $f(x) \geqslant g(x) \not\Longrightarrow \inf_x f(x) \geqslant \sup_x g(x)$, for example take $f = g$. Then unless f is a constant, we won't have $\inf_x f(x) \geqslant \sup_x g(x) = \sup_x f(x)$.

To prove Theorem 5.8, the following Lemma is needed.

Lemma 5.9. *Let $P \subseteq Q$ be partitions. Then $U(f, Q) \leqslant U(f, P)$ and $L(f, Q) \geqslant L(f, P)$.*

Proof. We prove the first claim. The proof for the second is similar.

Let $Q = \{x_0, x_1, \ldots, x_n\}$. Then as $P \subseteq Q$, we have $P = \{x_{n_0}, \ldots, x_{n_k}\}$ with $k \leqslant n$, and $n_0 = 0$, $n_k = n$. Now consider

$$\begin{aligned}
\left[\sup_{x \in [x_{n_{i-1}}, x_{n_i}]} f(x)\right](x_{n_i} - x_{n_{i-1}}) &= \left[\sup_{x \in [x_{n_{i-1}}, x_{n_i}]} f(x)\right](x_{n_i} - x_{n_i - 1}) \\
&\quad + \left[\sup_{x \in [x_{n_{i-1}}, x_{n_i}]} f(x)\right](x_{n_i - 1} - x_{n_i - 2}) \\
&\quad + \cdots + \left[\sup_{x \in [x_{n_{i-1}}, x_{n_i}]} f(x)\right](x_{n_{i-1}+1} - x_{n_{i-1}}) \\
&\geqslant \left[\sup_{x \in [x_{n_i - 1}, x_{n_i}]} f(x)\right](x_{n_i} - x_{n_i - 1}) \\
&\quad + \left[\sup_{x \in [x_{n_i - 1}, x_{n_i - 2}]} f(x)\right](x_{n_i - 1} - x_{n_i - 2}) \\
&\quad + \cdots + \left[\sup_{x \in [x_{n_{i-1}}, x_{n_{i-1}+1}]} f(x)\right](x_{n_{i-1}+1} - x_{n_{i-1}}). \\
&= \sum_{j = n_{i-1}+1}^{n_i} \left[\sup_{x \in [x_{j-1}, x_j]} f(x)\right](x_j - x_{j-1}).
\end{aligned} \tag{5.10}$$

Therefore we have

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^{k} \left[\sup_{x \in [x_{n_{i-1}}, x_{n_i}]} f(x)\right](x_{n_i} - x_{n_{i-1}}) \\
&\geqslant \sum_{i=1}^{k} \left\{\sum_{j = n_{i-1}+1}^{n_i} \left[\sup_{x \in [x_{j-1}, x_j]} f(x)\right](x_j - x_{j-1})\right\} \\
&= \sum_{l=1}^{n} \left[\sup_{x \in [x_{l-1}, x_l]} f(x)\right](x_l - x_{l-1}) = U(f, Q).
\end{aligned} \tag{5.11}$$

□

Proof. (of Theorem 5.8) We first show that $U(f, P) \geqslant L(f, Q)$ for any partitions P, Q. By Lemmas 5.5 and 5.9 we have

$$U(f, P) \geqslant U(f, P \cup Q) \geqslant L(f, P \cup Q) \geqslant L(f, Q). \tag{5.12}$$

Now we take infinmum over P:

$$U(f) = \inf_{P \text{ is a partition}} U(f, P) \geqslant \inf_P L(f, Q) = L(f, Q) \tag{5.13}$$

Finally we take supreme over Q:

$$L(f) = \sup_Q L(f, Q) \leqslant U(f). \tag{5.14}$$

Thus ends the proof. $\square$

Definition 5.10. *A bounded function on* $[a, b]$ *is said to be Riemann integrable if* $L(f) = U(f)$. *In this case we denote this common value by* $\int_a^b f(x)\,\mathrm{d}x$.

In the following examples we set $a, b \in \mathbb{R}$, $a < b$.

Example 5.11. The constant function $f(x) = c$ is integrable with $\int_a^b f(x)\,\mathrm{d}x = c\,(b - a)$.

Proof. Let $P = \{x_0, \ldots, x_n\}$ be an arbitrary partition. Then we have

$$U(f, P) = \sum_{j=1}^{n} \left[\sup_{[x_{j-1}, x_j]} f\right] |x_j - x_{j-1}| = \sum_{j=1}^{n} c\,|x_j - x_{j-1}| = c\left[\sum_{j=1}^{n} |x_j - x_{j-1}|\right] = c\,(b - a). \tag{5.15}$$

Similarly we have

$$L(f, P) = \sum_{j=1}^{n} \left[\inf_{[x_{j-1}, x_j]} f\right] |x_j - x_{j-1}| = c\,(b - a). \tag{5.16}$$

Since P is arbitrary, we have $U(f) = c\,(b - a) = L(f)$, therefore f is integrable with $\int_a^b f(x)\,\mathrm{d}x = c\,(b - a)$. $\square$

Example 5.12. The function $f(x) = x$ is integrable with $\int_a^b f(x)\,\mathrm{d}x = \frac{b^2 - a^2}{2}$.

Proof. Let $P = \{x_0, \ldots, x_n\}$ be an arbitrary partition. Then

$$U(f, P) = \sum_{j=1}^{n} \left[\sup_{[x_{j-1}, x_j]} f\right] |x_j - x_{j-1}| = \sum_{j=1}^{n} x_j\,(x_j - x_{j-1}); \tag{5.17}$$

$$L(f, P) = \sum_{j=1}^{n} x_{j-1}\,(x_j - x_{j-1}). \tag{5.18}$$

Now we notice that

$$U(f, P) + L(f, P) = \sum_{j=1}^{n} (x_j + x_{j-1})\,(x_j - x_{j-1}) = \sum_{j=1}^{n} (x_j^2 - x_{j-1}^2) = x_n^2 - x_{n-1}^2 = b^2 - a^2, \tag{5.19}$$

and

$$U(f, P) - L(f, P) = \sum_{j=1}^{n} (x_j - x_{j-1})^2. \tag{5.20}$$

Now if we take a particular $P_n^* = \left\{a, a + \frac{b - a}{n}, \ldots, a + \frac{n - 1}{n}\,(b - a), b\right\}$, we have

$$U(f) - L(f) \leqslant U(f, P_n^*) - L(f, P_n^*) = \frac{(b - a)^2}{n}. \tag{5.21}$$

As this holds for all $n \in \mathbb{N}$, we conclude $U(f) = L(f)$. Thus f is integrable.

To find the integral, we use:

$$U(f) + L(f, P) \leqslant U(f, P) + L(f, P) = b^2 - a^2 \Longrightarrow U(f) + L(f) \leqslant b^2 - a^2; \tag{5.22}$$

On the other hand

$$U(f, P) + L(f) \geqslant U(f, P) + L(f, P) = b^2 - a^2 \Longrightarrow U(f) + L(f) \geqslant b^2 - a^2. \tag{5.23}$$

Summarizing, we have

$$U(f) = L(f), \qquad U(f) + L(f) = b^2 - a^2, \tag{5.24}$$

therefore $\int_a^b f(x)\,\mathrm{d}x = \frac{b^2 - a^2}{2}$. $\square$

Remark 5.13. Note that similar to the proof of Theorem 5.8, the fact that it's the same P in

$$U(f, P) + L(f, P) = b^2 - a^2 \tag{5.25}$$

makes it incorrect to conclude directly

$$\inf_P U(f,P) + \sup_P L(f,P) = b^2 - a^2. \tag{5.26}$$

Example 5.14. Consider the Dirichlet function: $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$, then $f(x)$ is not integrable.

Proof. Let P be an arbitrary partition. We know that in every $[x_{i-1}, x_i]$, there is at least one rational number and one irrational number. That is $\sup_{[x_{i-1},x_i]} f = 1$, $\inf_{[x_{i-1},x_i]} f = 0$. Consequently

$$U(f,P) = 1, \qquad L(f,P) = 0 \tag{5.27}$$

for every partition. Therefore $U(f) = 1 \neq 0 = L(f)$, that is f is not integrable. □

Theorem 5.15. *A bounded function on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{5.28}$$

More practically, if we can find partitions P_n such that $U(f,P_n) - L(f,P_n) \longrightarrow 0$, then f is integrable and

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\longrightarrow\infty} U(f,P_n) = \lim_{n\longrightarrow\infty} L(f,P_n). \tag{5.29}$$

Proof.

- "Only if". Let f be integrable. Then $U(f) = L(f)$. By definition there are partitions P_1 and P_2 such that

$$U(f,P_1) < U(f) + \varepsilon/2; \qquad L(f,P_2) > L(f) - \varepsilon/2. \tag{5.30}$$

  Now set $P = P_1 \cup P_2$. We have, by Lemma 5.9,

$$U(f,P) \leqslant U(f,P_1) < U(f) + \varepsilon/2; \qquad L(f,P) \geqslant L(f,P_2) > L(f) - \varepsilon/2. \tag{5.31}$$

  As $U(f) = L(f)$, and $U(f,P) \geqslant L(f,P)$, it follows that

$$U(f,P) - L(f,P) < \varepsilon. \tag{5.32}$$

- "If". Take $\varepsilon = 1/n$. Then there is a partition P_n such that

$$U(f,P_n) - L(f,P_n) < \varepsilon < 1/n. \tag{5.33}$$

  As $U(f,P_n) \geqslant U(f) \geqslant L(f) \geqslant L(f,P_n)$, we have $0 \leqslant U(f) - L(f) < 1/n$ for any $n \in \mathbb{N}$. This can hold only when $U(f) = L(f)$.

- Proof of the "more practically" part. Assume that there are partitions P_n such that $U(f,P_n) - L(f, P_n) \longrightarrow 0$.

  First we show that f is integrable. For any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $U(f, P_{N+1}) - L(f, P_{N+1}) < \varepsilon$. Therefore f is integrable.

  Now we show that $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\longrightarrow\infty} U(f,P_n) = \lim_{n\longrightarrow\infty} L(f,P_n)$. Note that the existence of the two limits is not given, that is we need to show this existence.

  Consider the partition $Q_n = P_1 \cup \cdots \cup P_n$. Then we have

$$L(f,Q_1) \leqslant L(f,Q_2) \leqslant \cdots \leqslant L(f,Q_n) \leqslant \cdots \leqslant U(f,Q_1); \tag{5.34}$$

$$U(f,Q_1) \geqslant U(f,Q_2) \geqslant \cdots \geqslant U(f,Q_n) \geqslant \cdots \geqslant L(f,Q_1). \tag{5.35}$$

  Thus $\lim_{n\longrightarrow\infty} U(f,Q_n)$ and $\lim_{n\longrightarrow\infty} L(f,Q_n)$ exist.

  Since

$$U(f,P_n) - L(f,P_n) \geqslant U(f,Q_n) - L(f,Q_n) \tag{5.36}$$

  we have, by Squeeze Theorem

$$U(f,Q_n) - L(f,Q_n) \longrightarrow 0 \tag{5.37}$$

which, together with the definition of $U(f), L(f)$, gives us

$$\int_a^b f(x)\,\mathrm{d}x = U(f) = L(f) = \lim_{n \longrightarrow \infty} U(f, Q_n) = \lim_{n \longrightarrow \infty} L(f, Q_n). \tag{5.38}$$

Finally we show that $\lim_{n\longrightarrow\infty} U(f, P_n)$ and $\lim_{n\longrightarrow\infty} L(f, P_n)$ exist and equal the integral. Write

$$U(f) \leqslant U(f, P_n) \leqslant L(f, Q_n) + U(f, P_n) - L(f, P_n) \tag{5.39}$$

Taking limits on both sides and apply Squeeze Theorem gives the desired result. □

As we have seen in Example 5.12, establishing integrability through definition is surprisingly tricky, even for very simple functions. Also note that what we essentially did in Example 5.12 is just proving a special case of Theorem 5.15.

Theorem 5.16. *If f is continuous over $[a, b]$, then f is integrable.*

Proof. (Not required) Since f is continuous, f is uniformly continuous. For any $\varepsilon > 0$, there is $\delta > 0$ such that whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon/(b - a)$.

Now take $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$, and let $P = \left\{a, a + \frac{b-a}{n}, a + 2\,\frac{b-a}{n}, \ldots, b\right\}$. Then we have

$$|f(x) - f(y)| < \varepsilon/(b - a) \qquad \forall x, y \in \left[a + j\,\frac{b-a}{n}, a + (j+1)\,\frac{b-a}{n}\right]. \tag{5.40}$$

It follows that

$$M_j(f) - m_j(f) < \varepsilon/(b - a). \tag{5.41}$$

Thus we have

$$U(f) - L(f) \leqslant U(f, P) - L(f, P) < \frac{\varepsilon}{b-a}\,(b - a) = \varepsilon. \tag{5.42}$$

So f is integrable. □

When f is not continuous, it may or may not be integrable.

Example 5.17. Let $f = \begin{cases} 1 & x < 0 \\ 0 & x \geqslant 0 \end{cases}$. Prove that f is integrable over $[-1, 1]$.

Proof. Thanks to Theorem 5.15, all we need to show is that, for every ε, there exists a partition $P = \{x_0, \ldots, x_n\}$ such that $U(f, P) - L(f, P) < \varepsilon$.

Take $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Set $P = \left\{-1, -1 + \frac{1}{n}, \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Then on each subinterval we have $\sup f = \inf f$ except on $\left[-\frac{1}{n}, 0\right]$ we have

$$\sup_{\left[-\frac{1}{n}, 0\right]} f = 1, \qquad \inf_{\left[-\frac{1}{n}, 0\right]} f = 0. \tag{5.43}$$

Therefore we have

$$U(f, P) - L(f, P) = \left[\sup_{\left[-\frac{1}{n}, 0\right]} f - \inf_{\left[-\frac{1}{n}, 0\right]} f\right] \left|0 - \left(-\frac{1}{n}\right)\right| = \frac{1}{n} < \varepsilon. \tag{5.44}$$

Thus ends the proof. □

An example of a bounded function that is not integrable turns out has to be quite weird, thanks to the following result in real analysis.

Remark 5.18. In real analysis we will see that a bounded function is Riemann integrable if and only if the set of all the points where the function is not continuous has measure 0, that is the "size" of this set is 0. For the Dirichlet function in Example 5.14 this set is the whole interval $[a, b]$ since it is discontinuous everywhere (can you prove it?), the size of $[a, b] = b - a \neq 0$.

As a consequence, any bounded piecewise continuous function won't work as an example for non-integrable functions since there are only finitely many points of discontinuity, and a set of finitely many points always have measure ("size") 0.

On the other hand, once f is unbounded, it is never integrable according to Definition 5.10, because the supreme of f over at least one of the subinterval would be ∞, causing $U(f,P)-L(f,P)=\infty$ for every partition P. Similarly, Definition 5.10 does not cover the case when the interval $[a,b]$ is of infinite size (that is $a=-\infty$ or $b=\infty$ or both). We will see in Section 5.3 that there is a natural extension (improper integrals) of integrability to these situations.

5.1.2. Operations of integrals.

Theorem 5.19. (Arithmetics) *Let f and g be integrable functions on $[a,b]$ and let c be a real number. Then*

a) $c\,f$ is integrable and $\int_a^b (c\,f)(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$.

b) $f+g$ is integrable and $\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$.

c) $f\,g$ is integrable.

d) If $|g|$ is bounded below by a positive constant, then f/g is integrable.

Proof. We prove b), c) and leave a) as exercise (note that you need to consider the cases $c>,<,=0$).

- b). We try to apply Theorem 5.15. For any $\varepsilon>0$, since f,g are integrable, let P,Q be partitions such that

$$U(f,P)-L(f,P)<\varepsilon, \qquad U(g,Q)-L(g,Q)<\varepsilon. \tag{5.45}$$

Now consider the partition $P\cup Q$. We have

$$U(f,P\cup Q)-L(f,P\cup Q)<\varepsilon/2, \qquad U(g,P\cup Q)-L(g,P\cup Q)<\varepsilon/2. \tag{5.46}$$

Since we have

$$L(f,P\cup Q)+L(g,P\cup Q)\leqslant L(f+g,P\cup Q)\leqslant U(f+g,P\cup Q)\leqslant U(f,P\cup Q)+U(g,P\cup Q) \tag{5.47}$$

We conclude that $U(f+g,P\cup Q)-L(f+g,P\cup Q)<\varepsilon$.

- c). Since f,g are bounded, there is $M>0$ such that $|f|,|g|\leqslant M$. Now for any $\varepsilon>0$, let P_1 be a partition such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2\,M}$ and let P_2 be a partition such that $U(g,P_2)-L(g,P_2)<\frac{\varepsilon}{2\,M}$. Now set $P=P_1\cup P_2=\{x_0=a,\ldots,x_n=b\}$. We have

$$U(f,P)-L(f,P), U(g,P)-L(g,P)<\frac{\varepsilon}{2\,M}. \tag{5.48}$$

Let y,z be two points in $[x_{j-1},x_j]$. We have

$$\begin{aligned} |f(y)\,g(y)-f(z)\,g(z)| &\leqslant |f(y)\,(g(y)-g(z))|+|g(z)\,(f(y)-f(z))| \\ &\leqslant M\,[|g(y)-g(z)|+|f(y)-f(z)|] \\ &\leqslant M\,[(\sup g-\inf g)+(\sup f-\inf f)] \\ &= M\,[M_j(g)-m_j(g)]+M\,[M_j(f)-m_j(f)]. \end{aligned} \tag{5.49}$$

As this holds for every pair y,z, we conclude

$$M_j(f\,g)-m_j(f\,g)\leqslant M\,[M_j(g)-m_j(g)]+M\,[M_j(f)-m_j(f)]. \tag{5.50}$$

Thus

$$\begin{aligned} U(f\,g,P)-L(f\,g,P) &= \sum_{j=1}^{n} [M_j(f\,g)-m_j(f\,g)]\,|x_j-x_{j-1}| \\ &\leqslant M\left\{\sum_{j=1}^{n}[M_j(f)-m_j(f)]\,|x_j-x_{j-1}|+\sum_{j=1}^{n}[M_j(g)-m_j(g)]|x_j-x_{j-1}|\right\} \\ &\leqslant M\,\{[U(f,P)-L(f,P)]-[U(g,P)-L(g,P)]\} \\ &< \varepsilon. \end{aligned} \tag{5.51}$$

Integrability now follows from Theorem 5.15.

We will not spend time inventing tricks to prove d) here. But notice that all the above claims become trivial to prove in light of Remark 5.18. For example, if f, g are integrable, then their sets of discontinuity A_f, A_g have measure 0. But then clearly the set of discontinuity for $f+g$, fg, f/g are all just subsets of $A_f \cup A_g$ so they all must have measure 0 too. □

Remark 5.20. Note that there is no clear relation between $\int_a^b f(x) \quad g(x) \quad \mathrm{d}x$ and $\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right)$.[5.1] This is what makes many problems in science and engineering hard to study mathematically. The most famous among these problems is turbulence in fluid mechanics.

Theorem 5.21. (Composite function) *Let f be integrable over $[a,b]$ and g be continuous. Then $g \circ f$ is integrable of $[a,b]$.*

Proof. The proof requires uniform continuity and is omitted. □

Theorem 5.22. *Let f be a bounded function on $[a,b]$. Let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on both $[a,c]$ and $[c,b]$.*

Proof.

- "Only if". If

  $$U(f,P) - L(f,P) < \varepsilon \tag{5.52}$$

  then we can refine (add c) to get $\tilde{P} = P_1 \cup Q_1$.

  $$U(f,P_1) + U(f,Q_1) < L(f,P_1) + L(f,Q_1) + \varepsilon \tag{5.53}$$

  which is enough since we have $U(f,P_1) \geqslant L(f,P_1)$.

  This part is easy as any partitions P_1, Q_1 of $[a,c]$, $[c,b]$ can be combined into a partition of $[a,b]$. Application of Theorem 5.15 then yields the desired result.

- "If". This part is easy as an partition P can be refined to partitions P_1, Q_1.

□

Example 5.23. $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is integrable.

Proof. By Theorem 5.22, we only need to prove the integrability of f over $[0,b]$ for every $b > 0$ and $[a,0]$ for every $a < 0$. We prove the first case here, the proof for the second case is similar.

Thanks to Theorem 5.15, all we need to show is that, for every ε, there exists a partition $P = \{x_0 = 0, \ldots, x_n = b\}$ such that $U(f,P) - L(f,P) < \varepsilon$. Let $\delta = \frac{\varepsilon}{2}$. Since f is continuous on $[\delta, b]$, it is integrable there. Thus there is a partition $P_1 = \{\delta, \ldots, b\}$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}. \tag{5.54}$$

Now set $P = \{0\} \cup P_1$. Then we have

$$U(f,P) = \left[\sup_{[0,\delta]} f\right] \delta + U(f,P_1) = \delta + U(f,P_1); \tag{5.55}$$

$$L(f,P) = \left[\inf_{[0,\delta]} f\right] \delta + L(f,P_1) = L(f,P_1). \tag{5.56}$$

Thus

$$U(f,P) - L(f,P) = \delta + U(f,P_1) - L(f,P_1) < \varepsilon. \tag{5.57}$$

Thus ends the proof. □

5.1. Or more precisely, between the "scale-free" quantities $\frac{1}{b-a}\int_a^b f(x)\,g(x)\mathrm{d}x$ and $\left[\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right]\left[\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right]$.

5.2. The Fundamental Theorem of Calculus.

As we have seen in Example 5.12, getting $\int_a^b f(x)\,\mathrm{d}x$ by definition is very complicated even for simple functions like $f(x)=x$. Fortunately, the following theorems makes this task much easier through connecting integration to (the much simpler) differentation. Let's work on one more example to appreciate this better.

Example 5.24. Use definition to calculate $\int_a^b \cos x\,\mathrm{d}x$.

First since $\cos x$ is bounded and continuous on $[a,b]$, it is integrable. For $n\in\mathbb{N}$ let $P_n=\{x_0=a,\ldots,x_n=b\}$ with $x_j-x_{j-1}=\frac{b-a}{n}$. Then we calculate

$$\begin{aligned}\frac{b-a}{n}\sum_{j=1}^{n}[\cos x_j] &= \frac{b-a}{n}\sum_{j=1}^{n}\cos\left[a+\frac{j}{n}(b-a)\right]\\ &= \frac{b-a}{n}\sum_{j=1}^{n}\frac{\sin\frac{b-a}{2n}\cos\left[a+\frac{j}{n}(b-a)\right]}{\sin\frac{b-a}{2n}}\\ &= \frac{b-a}{n}\sum_{j=1}^{n}\frac{\sin\left[a+\left(\frac{j}{n}+\frac{1}{2n}\right)(b-a)\right]-\sin\left[a+\left(\frac{j}{n}-\frac{1}{2n}\right)(b-a)\right]}{2\sin\frac{b-a}{2n}}\\ &= \frac{\frac{b-a}{2n}}{\sin\frac{b-a}{2n}}\left[\sin\left(b+\frac{b-a}{2n}\right)-\sin\left(a+\frac{b-a}{2n}\right)\right]. \end{aligned}\tag{5.58}$$

Taking $n\longrightarrow\infty$ we conclude that (recall that $\frac{\sin x}{x}\longrightarrow 1$ as $x\longrightarrow 0$)

$$\lim_{n\longrightarrow\infty}\frac{b-a}{n}\sum_{j=1}^{n}[\cos x_j]=\sin b-\sin a.\tag{5.59}$$

Now consider an arbitrary partition Q. Set $P=Q\cup P_n$. Then on each subinterval inside $[x_{j-1},x_j]$ we have

$$|\sup(\cos x)-\cos x_j|,|\inf(\cos x)-\cos x_j|\leqslant\frac{b-a}{n}\tag{5.60}$$

which leads to

$$U(\cos x,Q)\geqslant U(\cos x,P)\geqslant\frac{b-a}{n}\sum_{j=1}^{n}[\cos x_j]-\frac{(b-a)^2}{n}\tag{5.61}$$

and

$$L(\cos x,Q)\leqslant L(\cos x,P)\leqslant\frac{b-a}{n}\sum_{j=1}^{n}[\cos x_j]+\frac{(b-a)^2}{n}.\tag{5.62}$$

Taking limit $n\longrightarrow\infty$ we conclude

$$U(\cos x,Q)\geqslant\sin b-\sin a\geqslant L(\cos x,Q).\tag{5.63}$$

As this holds for all partitions Q, we have $U(\cos x)\geqslant\sin b-\sin a\geqslant L(\cos x)$.

Finally because $\cos x$ is integrable, we must have $\int_a^b\cos x\,\mathrm{d}x=U(\cos x)=L(\cos x)=\sin b-\sin a$.

Remark 5.25. In the above argument we used the fact that $\cos x$ is differentiable. It can be proved that in general, if $f(x)$ is integrable over $[a,\ b]$, then $U(f,\ P_n)\longrightarrow\int_a^b f(x)\ \mathrm{d}x$ when $n\longrightarrow\infty$. Here $P_n=\left\{a,\ a+\frac{b-a}{n},\ \ldots,\ a+\frac{n-1}{n}(b-a),\ b\right\}$ as in the above example. More generally, any partition with $\max_j|x_j-x_{j-1}|\longrightarrow 0$ would do.

5.2.1. The theorems.

Definition 5.26. *Let f be a real function on an interval I. A function F on I is called an antiderivative of f on I if $F'(x)=f(x)$ for all $x\in I$. F is also called an indefinite integral of f, denoted by $\int f(x)\,\mathrm{d}x$.*

Lemma 5.27. *If F is an antiderivative so is $F+C$. And all antiderivatives are of this form.*

Proof. If F is an antiderivative then by definition $F'(x)=f(x)$. This means $(F+C)'=f(x)$ and therefore $F+C$ is also an antiderivative.

On the other hand, if G is also an antiderivative, then $G'(x)=f(x)=F'(x)\Longrightarrow(G-F)'=0\Longrightarrow G-F$ is a constant. Call this constant C we reach $G(x)=F(x)+C$. □

Theorem 5.28. (FTC 1st Version) *Let f be integrable on $[a,b]$. If F is continuous on $[a,b]$ and is an antiderivative of f on (a,b), then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{5.64}$$

Remark 5.29. Note that if F is an antiderivative of f on (a,b) then F is differentiable on (a,b) and therefore continuous on (a,b). So what the condition "F is continuous on $[a,b]$" really is about are continuity at the end points a,b.

Proof. Take any $\varepsilon > 0$. As f is integrable there is a partition $P = \{x_0, \ldots, x_n\}$ such that $U(f,P) - L(f,P) < \varepsilon$ which leads to $\int_a^b f(x) - \varepsilon < L(f,P) \leqslant U(f,P) < \int_a^b f(x) + \varepsilon$

Now since F is continuous on $[a,b]$ and $F' = f$ on (a,b), by mean value theorem we have

$$F(x_i) - F(x_{i-1}) = f(\xi_i)\,(x_i - x_{i-1}) \tag{5.65}$$

for $\xi_i \in (x_{i-1}, x_i)$. Recalling the definitions of $U(f,P), L(f,P)$ we have

$$L(f,P) \leqslant \sum_{i=1}^n F(x_i) - F(x_{i-1}) \leqslant U(f,P) \tag{5.66}$$

which leads to

$$\int_a^b f(x) - \varepsilon \leqslant F(b) - F(a) \leqslant \int_a^b f(x) + \varepsilon. \tag{5.67}$$

This holds for any $\varepsilon > 0$, consequently we must have $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$. ☐

Remark 5.30. The assumptions are all necessary. In particular we have to assume f to be integrable since otherwise $\int_a^b f(x)\,\mathrm{d}x$ is not defined. It can indeed happen that $f(x) = F'(x)$ is not integrable. For example $F(x) = x \sin\frac{1}{x}$ and $f(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ which is not bounded and therefore is not integrable.

Example 5.31. We compute $\int_a^b \cos x\,\mathrm{d}x$ again.

Since $\cos x$ is bounded and continuous on $[a,b]$ it is integrable. We know that $\sin x$ is one antiderivative of $\cos x$. As $\sin x$ is continuous on $[a,b]$ we can apply FTC 1st version to get

$$\int_a^b \cos x\,\mathrm{d}x = \sin b - \sin a. \tag{5.68}$$

Theorem 5.32. (FTC 2nd Version) *Let f be integrable on $[a,b]$. Then*

$$G(x) := \int_a^x f(t)\,\mathrm{d}t \tag{5.69}$$

is continuous on $[a,b]$. Furthermore, if f is continuous at a point $x_0 \in (a,b)$, then G is differentiable at x_0 and

$$G'(x_0) = f(x_0). \tag{5.70}$$

Proof. First show G continuous. Since f is integrable, it is bounded. Let $M > 0$ be such that $|f(x)| \leqslant M$ for all $x \in [a,b]$.

Now set $x_0 \in [a,b]$. For any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{M}$. We have, for all $x_0 \leqslant x < x_0 + \delta$,

$$|G(x) - G(x_0)| = \left|\int_{x_0}^x f(t)\,\mathrm{d}t\right| \leqslant M\,(x - x_0) < \varepsilon. \tag{5.71}$$

Similarly we can show that the same holds for $x_0 - \delta < x \leqslant x_0$. Thus for all $|x - x_0| < \delta$ we have $|G(x) - G(x_0)| < \varepsilon$ that is G is continuous at x_0.

Now we prove $G'(x_0) = f(x_0)$ when f is continuous at x_0. For any $\varepsilon > 0$, since f is continuous at x_0 there is $\delta > 0$ such that when $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$. Now for such x we have

- either $x > x_0$, in this case

$$\left|\frac{G(x) - G(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^x (f(t) - f(x_0))\,\mathrm{d}t\right| < \varepsilon, \tag{5.72}$$

- or $x < x_0$, in this case

$$\left|\frac{G(x) - G(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\left[-\int_x^{x_0} (f(t) - f(x_0))\,\mathrm{d}t\right]\right| < \varepsilon, \tag{5.73}$$

Therefore

$$\lim_{x \to x_0} \frac{G(x) - G(x_0)}{x - x_0} = f(x_0) \tag{5.74}$$

and the proof ends. □

Remark 5.33. In the above we used the following facts:

1. If $f \leqslant g$, then $\int_a^b f\,\mathrm{d}x \leqslant \int_a^b g\,\mathrm{d}x$.
2. $\left|\int_a^b f\,\mathrm{d}x\right| \leqslant \int_a^b |f|\,\mathrm{d}x$.

They can be easily proved using definition. The proofs are left as exercises.

Remark 5.34. Traditionally, $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ means $\displaystyle-\int_b^a f(x)\,\mathrm{d}x$ when $a > b$. Note that this notation is reasonable, for example if $x < x_0$, we have

$$\int_a^x f(t)\,\mathrm{d}t - \int_a^{x_0} f(t)\,\mathrm{d}t = -\int_x^{x_0} f(t)\,\mathrm{d}t \tag{5.75}$$

which now can be simply written as

$$\int_{x_0}^x f(t)\,\mathrm{d}t. \tag{5.76}$$

Thus we can write

$$\int_a^x f(t)\,\mathrm{d}t - \int_a^{x_0} f(t)\,\mathrm{d}t = \int_{x_0}^x f(t)\,\mathrm{d}t \tag{5.77}$$

for all x, x_0, whichever is bigger.

Obviously, if $a = b$, then

$$\int_a^b f(x)\,\mathrm{d}x = 0 \tag{5.78}$$

for any integrable function f.

Remark 5.35. Ideally one would like to have a theorem stating

$$f(x) \text{ integrable} \Longrightarrow G(x) := \int_a^x f(t)\,\mathrm{d}t \text{ differentiable and } G'(x) = f(x). \tag{5.79}$$

However this is not true. An example is $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$, for which $G(x) = \begin{cases} 0 & x < 0 \\ x & x \geqslant 0 \end{cases}$ which is not differentiable at $x = 0$.

It may also happen that $\int_a^x f(t)\,\mathrm{d}t$ is differentiable at some point x_0 but the derivative is not $f(x_0)$. For example let $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. Set $a = -1$. Then

$$G(x) = \int_a^x f(t)\,\mathrm{d}t = 0 \tag{5.80}$$

for all x but $G'(0) = 0 \neq f(0) = 1$.[5.2]

5.2.2. Applications.

Theorem 5.36. (Integration by parts) *If u, v are continuous on $[a, b]$ an differentiable on (a, b), and if u', v' are integrable on $[a, b]$, then*

$$\int_a^b u(x)\,v'(x)\,\mathrm{d}x = u(b)\,v(b) - u(a)\,v(a) - \int_a^b u'(x)\,v(x)\,\mathrm{d}x. \tag{5.81}$$

5.2. Clearly this counterexample is very artificial. There is a reason behind this. It turns out that, if $f(x)$ is integrable over $[a, b]$, then the derivative of $\int_a^x f(t)\,\mathrm{d}t$ must coincide with $f(x)$ for "almost all" $x \in (a, b)$. This is the so-called Lebesgue's Differentiation Theorem.

Proof. Let $F = u\,v$. Then $F' = u\,v' + u'\,v$. Since u, v are continuouson $[a, b]$, they are also integrable on $[a, b]$. Together with integrability of u', v' we conclude F' is integrable (Theorem 5.19) on $[a, b]$. Application of the first version of FTC gives the desired result. □

Example 5.37. Calculate

$$\int_0^1 x\,e^x\,\mathrm{d}x. \tag{5.82}$$

Solution. We set $v = e^x, u = x$. Then we have

$$\begin{aligned}\int_0^1 x\,e^x\,\mathrm{d}x &= \int_0^1 u\,v'\,\mathrm{d}x \\ &= u(1)\,v(1) - u(0)\,v(0) - \int_0^1 v\,u'\,\mathrm{d}x \\ &= e - \int_0^1 e^x\,\mathrm{d}x \\ &= e - e^x|_{x=0}^{x=1} \\ &= 1. \end{aligned} \tag{5.83}$$

Example 5.38. Taylor expansion with integral remainder.

We can obtain Taylor expansion using integration by parts.

$$\begin{aligned} f(x) - f(a) &= \int_a^x f'(t)\,\mathrm{d}t \\ &= [t\,f'(t)]_a^x - \int_a^x t\,f''(t)\,\mathrm{d}t \\ &= f'(a)\,(x-a) + x\,[f'(x) - f'(a)] - \int_a^x t\,f''(t)\,\mathrm{d}t \\ &= f'(a)\,(x-a) + \int_a^x (x-t)\,f''(t)\,\mathrm{d}t \\ &= f'(a)\,(x-a) - \int_a^x f''(t) \left[\frac{(x-t)^2}{2}\right]'\mathrm{d}t \\ &= f'(a)\,(x-a) - \left\{ f''(t)\,\frac{(x-t)^2}{2} \right\}_{t=a}^{t=x} + \int_a^x \frac{(x-t)^2}{2}\,f'''(t)\,\mathrm{d}t \\ &= f'(a)\,(x-a) + \frac{f''(a)}{2}\,(x-a)^2 + \int_a^x \frac{(x-t)^2}{2}\,f'''(t)\,\mathrm{d}t \\ &= \cdots = f'(a)\,(x-a) + \frac{f''(a)}{2}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^2 + \int_a^x \frac{(x-t)^3}{6}\,f^{(4)}(t)\,\mathrm{d}t \\ &\vdots \\ &= \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}\,(x-a)^m + \int_a^x \frac{(x-t)^n}{n!}\,f^{(n+1)}(t)\,\mathrm{d}t. \end{aligned} \tag{5.84}$$

The remainder $\int_a^x \frac{(x-t)^n}{n!}\,f^{(n+1)}(t)\,\mathrm{d}t$ is called "integral remainder". One can show that if $f^{(n+1)}(t)$ is continuous, then there is $\xi \in (a, x)$ such that

$$\int_a^x \frac{(x-t)^n}{n!}\,f^{(n+1)}(t)\,\mathrm{d}t = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-a)^{n+1} \tag{5.85}$$

which is exactly the Lagrange remainder. The disadvantage of the Lagrange remainder is that

1. We have no knowledge of where ξ exactly is;
2. The dependence of ξ on x may be rough. For example, we can differentiate the integral remainder but not the Lagrange remainder (due to $\xi(x)$ may not be differentiable).

Therefore in analysis one should try to use integral remainder whenever possible.

Theorem 5.39. (Change of variables) *Let u be differentiable and assume u' is integrable on $[a,b]$. If f is continuous on $I := u([a,b])$, then*

$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x. \tag{5.86}$$

Remark 5.40. In the case $u(b) < u(a)$, the integral is understood as

$$\int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x = -\int_{u(b)}^{u(a)} f(x)\, \mathrm{d}x. \tag{5.87}$$

Proof. We notice that, if we define $F(x) = \int_{u(a)}^{x} f(t)\, \mathrm{d}t$, then $F'(x) = f(x)$ and it follows from FTC Version 1 that

$$\int_{u(a)}^{u(b)} f(t)\, \mathrm{d}t = F(u(b)) - F(u(a)); \tag{5.88}$$

On the other hand, if we set

$$G(t) := F(u(t)) \tag{5.89}$$

then by Chain rule

$$G'(t) = \frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = F'(u(t))\, u'(t) = f(u(t))\, u'(t). \tag{5.90}$$

Note that the last equality is a result of FTC Version 2 and only holds because f is continuous.

Next we check that $f(u(t))\, u'(t)$ is integrable: $f(x), u(t)$ continuous $\Longrightarrow$ $f(u(t))$ continuous $\Longrightarrow f(u(t))$ integrable $\Longrightarrow$ $f(u(t))\, u'(t)$ integrable thanks to Theorem 5.19 since $u'(t)$ is integrable.

Finally applying FTC Version 1 to G we have

$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = G(b) - G(a) = F(u(b)) - F(u(a)). \tag{5.91}$$

and the proof ends. □

Remark 5.41. Note that we **don't** need u to be one-to-one!

Example 5.42. Calculate

$$\int_0^{\pi/4} \frac{\mathrm{d}x}{\cos x}. \tag{5.92}$$

Solution. Set $y = \sin x$. We see that $[0, \pi/4]$ is mapped to $[\sin 0, \sin(\pi/4)] = \left[0, \frac{\sqrt{2}}{2}\right]$.

Now calculate

$$\begin{aligned} \int_0^{\pi/4} \frac{\mathrm{d}x}{\cos x} &= \int_0^{\pi/4} \frac{\cos x\, \mathrm{d}x}{\cos^2 x} \\ &= \int_0^{\pi/4} \frac{\cos x\, \mathrm{d}x}{1 - \sin^2 x} \\ &= \int_0^{\sqrt{2}/2} \frac{\mathrm{d}y}{1 - y^2} \\ &= -\frac{1}{2} \int_0^{\sqrt{2}/2} \left[\frac{\mathrm{d}y}{y-1} - \frac{\mathrm{d}y}{y+1}\right] \\ &= -\frac{1}{2} [\ln|y-1| - \ln|y+1|]_{y=0}^{y=\sqrt{2}/2} \\ &= \frac{1}{2} \ln\left(\frac{y+1}{1-y}\right)_{y=0}^{y=\sqrt{2}/2} \\ &= \frac{1}{2} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right). \end{aligned} \tag{5.93}$$

5.3. Improper Riemann Integration.

Recall that Definition 5.10 cannot deal with unbounded functions or unbounded intervals. However for some functions there is a reasonable way to define integrals.

Definition 5.43. *Let $a<b$ where a,b may be extended real numbers. Let $f\colon (a,b)\mapsto \mathbb{R}$. Then f is said to be improperly integrable on (a,b) if*

a) *f is locally integrable: For any $c,d\in\mathbb{R}$, $a<c<d<b$, f is integrable on $[c,d]$, and*

b) *$\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,\mathrm{d}x\right)$ exists and is finite. Call this limit the improper Riemann integral of f over (a,b), denote it by*

$$\int_a^b f(x)\,\mathrm{d}x. \tag{5.94}$$

We state without proof the following theorem which guarantees the consistency of the above definition.

Theorem 5.44.

a) *If f is integrable on (a,b), then it is improperly integrable and its improper integral equals its integral.*

b) *If f is improperly integrable on (a,b), then the order of limit taking does not matter:*

$$\int_a^b f(x)\,\mathrm{d}x=\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,\mathrm{d}x\right)=\lim_{d\longrightarrow b-}\left(\lim_{c\longrightarrow a+}\int_c^d f(x)\,\mathrm{d}x\right). \tag{5.95}$$

Proof. Not required. □

Example 5.45. Show that $f(x)=x^{-1/2}$ is improperly integrable on $(0,1]$.

Proof. For any $0<c<d<1$ f is continuous on $[c,d]$ so f is locally integrable. We calculate

$$\int_c^d f(x)\,\mathrm{d}x=2\,x^{1/2}|_{x=c}^{x=d}=2\left[d^{1/2}-c^{1/2}\right]. \tag{5.96}$$

Then

$$\lim_{c\to 0+}\left(\lim_{d\to 1-}\int_c^d f(x)\,\mathrm{d}x\right)=\lim_{c\to 0+}2\left[1-c^{1/2}\right]=2. \tag{5.97}$$

So f is improperly integrable on $(0,1]$ and its improper integral over $(0,1)$ is 2. □

Example 5.46. Show that $f(x)=x^{-1}$ is not improperly integrable on $(0,1)$.

Proof. For any $0<c<d<1$ f is continuous on $[c,d]$ so f is locally integrable. We have

$$\int_c^d x^{-1}\,\mathrm{d}x=\ln d-\ln c. \tag{5.98}$$

Then

$$\lim_{c\to 0+}\left(\lim_{d\to 1-}\int_c^d f(x)\,\mathrm{d}x\right)=\lim_{c\longrightarrow 0+}-\ln c=\infty \tag{5.99}$$

The limit is not finite so f is not improperly integrable on $(0,1)$. □

Example 5.47. Show that $f(x)=\begin{cases}1 & x \text{ rational}\\ 0 & x \text{ irrational}\end{cases}$ is not improperly integrable on $(0,1)$.

Proof. For $0<c<d<1$, we know that $f(x)$ is not integrable on (c,d). So f is not locally integrable and cannot be improperly integrable. □

Theorem 5.48. (Arithmetics) *Let f, g be improperly integrable on (a, b). Then*

a) *For any $c \in \mathbb{R}$, $c f$ is improperly integrable on (a, b) with $\int_a^b c f(x)\,\mathrm{d}x = c \int_a^b f(x)\,\mathrm{d}x$.*

b) *$f + g$ is improperly integrable on (a, b) with $\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$.*

c) *If g is bounded and locally integrable, and $|f|$ is improperly integrable, then $f g$ is improperly integrable.*

Instead of proving the theorem, we discuss briefly part c).

- Note that we do not need g to be improperly integrable on (a, b). This relaxation should become natural in light of the following example:

  Take $a = 0, b = \infty$. Take $g = 1$. Then g is not improperly integrable on $(0, \infty)$ but clearly if f is improperly integrable then so is $f g$.

- The (stronger) requirement that $|f|$ (instead of f) being improperly integrable is necessary, as can be seen from the following example:

  Consider $a = \pi, b = \infty$ and $f(x) = \frac{\sin x}{x}$. Then for any $1 < c < d < \infty$, we have

$$\begin{aligned} \int_c^d \frac{\sin x}{x}\,\mathrm{d}x &= \int_c^d \frac{1}{x}(-\cos x)'\,\mathrm{d}x \\ &= \left[\frac{\cos d}{d} - \frac{\cos c}{c}\right] + \int_c^d \frac{\cos x}{x^2}\,\mathrm{d}x. \end{aligned} \tag{5.100}$$

  Now the limit of the first term obviously exists. For the second term, since $\frac{\cos x}{x^2}$ is continuous on $[\pi, d]$ for any d, we have

$$\lim_{d \longrightarrow \infty}\left[\lim_{c \longrightarrow \pi}\int_c^d \frac{\cos x}{x^2}\,\mathrm{d}x\right] = \lim_{d \longrightarrow \infty}\int_\pi^d \frac{\cos x}{x^2}\,\mathrm{d}x. \tag{5.101}$$

  It turns out that this limit exists: Set $F(d) := \int_\pi^d \frac{\cos x}{x^2}\,\mathrm{d}x$, we only need to show that $\lim_{d \longrightarrow \infty} F(d)$ exists and is finite. It suffices to show that for any sequence $d_n \longrightarrow \infty$, $\lim_{n \longrightarrow \infty} F(d_n)$ exists and is finite. For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $d_n > \varepsilon^{-1}$ for all $n > N$. Then for any $n, m > N$, we have (assume $m > n$)

$$\begin{aligned} |F(d_m) - F(d_n)| &= \left|\int_{d_n}^{d_m} \frac{\cos x}{x^2}\,\mathrm{d}x\right| \\ &\leqslant \int_{d_n}^{d_m} \frac{1}{x^2}\,\mathrm{d}x \\ &= \left|\frac{1}{d_n} - \frac{1}{d_m}\right| \\ &< \max\left\{\frac{1}{d_n}, \frac{1}{d_m}\right\} < \varepsilon. \end{aligned} \tag{5.102}$$

  Thus $F(d_n)$ is Cauchy and therefore converges to a finite limit.

  Therefore $f(x)$ is improperly integrable over (π, ∞). Now take $g(x) = \begin{cases} 1 & \sin x > 0 \\ -1 & \sin x \leqslant 0 \end{cases}$. We have $f(x)\,g(x) = \frac{|\sin x|}{x}$. We have

$$\begin{aligned} \int_\pi^{n\pi} f(x)\,g(x)\,\mathrm{d}x &= \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,\mathrm{d}x \\ &\geqslant \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|\,\mathrm{d}x \\ &= \frac{2}{\pi}\sum_{k=1}^{n-1} \frac{1}{(k+1)}. \end{aligned} \tag{5.103}$$

This sum tends to ∞ as $n \longrightarrow \infty$. So we know

$$\int_{\pi}^{n\pi} f(x)\, g(x)\, \mathrm{d}x = \infty \tag{5.104}$$

which means even if

$$\lim_{d \longrightarrow \infty} \int_{\pi}^{d} f(x)\, g(x)\, \mathrm{d}x \tag{5.105}$$

exists, it cannot be finite. Therefore $f g$ is not improperly integrable on (π, ∞).

Remark 5.49. Recall that the function $\frac{\sin x}{x}$ is integrable on $(0, 1]$ ($[0, 1]$ if we further define its value at 0 to be 1). Therefore $\frac{\sin x}{x}$ is improperly integrable on $(0, \infty)$.