4. Differentiation

4.1. Derivatives.

4.1.1. Definition.

Definition 4.1. *Let f be a real function. At a point x_0 inside its domain, if the limit*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{4.1}$$

exists, we say f is differentiable at x_0, and call the limit its derivative at x_0, denoted $f'(x_0)$. If the limit does not exist, we say f is not differentiable at x_0. If f is differentiable at all $x \in E$ where $E \subseteq \mathbb{R}$, we say f is differentiable on E. If f is differentiable at every point of its domain, we say f is differentiable.

Remark 4.2. Equivalently, one can define differentiability through the limit

$$\lim_{\delta \longrightarrow 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta}. \tag{4.2}$$

That is f is differentiable at x_0 if the above limit exists.

Remark 4.3. Recall that in the definition of limits, we require $0 < |x - x_0|$. This is crucial in the limit (4.1) since at $x = x_0$ we have $\frac{0}{0}$.

Example 4.4. Let $f(x) = x^n$ with $n \in \mathbb{N} \cup \{0\}$. Study its differetiability.
Solution. When $n = 0$ we have $f(x) = 1$ for all x. Then for every $x_0 \in \mathbb{R}$,

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} 0 = 0. \tag{4.3}$$

So $(x^0)' = 0$.

When $n = 1$ we have $f(x) = x$. For every $x_0 \in \mathbb{R}$,

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} \frac{x - x_0}{x - x_0} = \lim_{x \longrightarrow x_0} 1 = 1. \tag{4.4}$$

So $(x^1)' = 1$.

For $n \geqslant 2$ we have for every $x_0 \in \mathbb{R}$,

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \longrightarrow x_0} (x^{n-1} + x^{n-2} x_0 + \cdots + x\, x_0^{n-2} + x_0^{n-1}) = n\, x_0^{n-1}. \tag{4.5}$$

Therefore $(x^n)' = n\, x^{n-1}$.

From the above it is easy to obtain

Lemma 4.5. (Derivative of constant functions) *Let $f(x) = a$ for all x in its domain. Then $f'(x) = 0$.*

Proof. Left as exercise. □

Lemma 4.6. (Differentiable functions are continuous) *If $f(x)$ is differentiable at x_0, then $f(x)$ is continuous at x_0.*

Proof. Since $f(x)$ is differentiable at x_0, we have by definition

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \in \mathbb{R} \tag{4.6}$$

Now write

$$f(x) = f(x_0) + (x - x_0)\,\frac{f(x) - f(x_0)}{x - x_0} \tag{4.7}$$

and take limit $x \longrightarrow x_0$, we have

$$\lim_{x \to x_0} f(x) = f(x_0) + \left[\lim_{x \to x_0} x - x_0\right]\left[\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right] = f(x_0) + 0 \cdot L = f(x_0) \tag{4.8}$$

Therefore $f(x)$ is continuous at x_0. □

Remark 4.7. Note that one can also prove using definition as follows. Since $f(x)$ is differentiable at x_0, we have by definition

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \in \mathbb{R} \tag{4.9}$$

Take $\delta_1 > 0$ such that for all $0 < |x - x_0| < \delta_1$,

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| < 1 \Longrightarrow |f(x) - f(x_0)| < (|L| + 1)\,|x - x_0|. \tag{4.10}$$

Now for any $\varepsilon > 0$, take $\delta = \min\left\{\delta_1, \frac{\varepsilon}{|L| + 1}\right\}$. We have, for all $0 < |x - x_0| < \delta$,

$$|f(x) - f(x_0)| < (|L| + 1)\,|x - x_0| < (|L| + 1)\,\delta \leqslant \varepsilon. \tag{4.11}$$

4.1.2. Operations of derivatives.

Theorem 4.8. (Arithmetics of derivatives) *Let f, g be differentiable at x_0. Then*

a) *$f \pm g$ is differentiable at x_0 with $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.*

b) **(Leibniz rule)** *fg is differentiable at x_0 with $(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$.*

c) *If $g(x_0) \neq 0$, then f/g is differentiable at x_0 with*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{g(x_0)^2}. \tag{4.12}$$

Proof.

a) We have

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}. \tag{4.13}$$

Since

$$\lim_{x \to x_0}\left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}\right] = f'(x_0) + g'(x_0) \tag{4.14}$$

The limit

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} \tag{4.15}$$

also exists and equals $f'(x_0) + g'(x_0)$. The case $f - g$ can be proved similarly.

b) We have

$$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f(x)\,\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\,\frac{f(x) - f(x_0)}{x - x_0}. \tag{4.16}$$

Since

$$\lim_{x \to x_0} f(x) = f(x_0); \quad \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0); \quad \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \tag{4.17}$$

we reach

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f(x_0)\,g'(x_0) + f'(x_0)\,g(x_0). \tag{4.18}$$

c) We only prove the last one. In light of b), it suffices to prove

$$\left(\frac{1}{g}\right)' = -\frac{g'(x_0)}{g^2(x_0)}. \tag{4.19}$$

Write

$$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = -\frac{\frac{g(x) - g(x_0)}{x - x_0}}{g(x)\, g(x_0)}. \tag{4.20}$$

Note that both the denominator and the numerator have limits, and furthermore the limit of the denominator is not 0. So we have the limit of the ratio exists and

$$\lim_{x \longrightarrow x_0} \left[-\frac{\frac{g(x) - g(x_0)}{x - x_0}}{g(x)\, g(x_0)} \right] = -\frac{\lim_{x \longrightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0}}{\lim_{x \longrightarrow x_0} g(x)\, g(x_0)} = -\frac{g'(x_0)}{g(x_0)^2}. \tag{4.21}$$

Thus ends the proof. □

Example 4.9. Compute $(x^{-n})'$ for $n \in \mathbb{N}$.
Solution. Note that the domain of x^{-n} is $\mathbb{R}\backslash\{0\}$. For any $x_0 \in \mathbb{R}\backslash\{0\}$ we have $x_0^n \neq 0$, so

$$(x^{-n})'(x_0) = \left(\frac{1}{x^n}\right)'(x_0) = -\frac{(x^n)'|_{x = x_0}}{(x^n)^2|_{x = x_0}} = -n\, x_0^{-n-1}. \tag{4.22}$$

So $(x^{-n})' = -n\, x^{-n-1}$.

Theorem 4.10. (Chain rule) *If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composite function $g \circ f$ is differentiable at x_0 and satisfy*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0). \tag{4.23}$$

Proof. Set

$$h(y) := \begin{cases} \dfrac{g(y) - g(f(x_0))}{y - f(x_0)} & y \neq f(x_0) \\ g'(f(x_0)) & y = f(x_0) \end{cases}. \tag{4.24}$$

Then we have $h(y)$ satisfying $\lim_{y \longrightarrow f(x_0)} h(y) = h(f(x_0))$.

Now write

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = h(f(x))\, \frac{f(x) - f(x_0)}{x - x_0}. \tag{4.25}$$

By Lemma 4.6 we have $\lim_{x \longrightarrow x_0} f(x) = f(x_0)$. Thus taking limit of both sides of (4.25) we reach

$$\lim_{x \longrightarrow x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \left(\lim_{x \longrightarrow x_0} h(f(x))\right) \left(\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}\right) = h(f(x_0))\, f'(x_0) \tag{4.26}$$

and the proof ends. □

Remark 4.11. Naturally one may want to prove through

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}\, \frac{f(x) - f(x_0)}{x - x_0} \tag{4.27}$$

and try to show

$$\lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = g'(f(x_0)). \tag{4.28}$$

However this does not work because it may happen that $f(x) - f(x_0) = 0$. The above trick overcomes this difficulty.

Theorem 4.12. (Derivative of inverse function) *Let f be differentiable at x_0 with $f'(x_0) \neq 0$. Then if f has an inverse function g, then g is differentiable at $y_0 = f(x_0)$ and satisfies $g'(f(x_0)) = 1/f'(x_0)$ or equivalently $g'(y_0) = 1/f'(g(y_0))$.*

Proof. Since f has an inverse function, f is either strictly increasing or strictly decreasing. Furthermore g is continuous, and also strictly increasing or decreasing.

Let $y_0 = f(x_0)$. We compute

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{g(y) - g(y_0)}{f(g(y)) - f(g(y_0))} = \left(\frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)}\right)^{-1}. \tag{4.29}$$

Note that as f, g are both strictly increasing/decreasing, all the denominators in the above formula are nonzero. To show that the limit exists, we recall that $\lim F(x)$ exists at x_0 if for all $x_n \longrightarrow x_0$ the limit of $F(x_n)$ exists.

Take $y_n \longrightarrow y_0$. By continuity of g we have $g(y_n) \longrightarrow g(y_0)$. The differentiability of f at $g(y_0)$, that is the existence of the limit $\lim_{x \longrightarrow g(y_0)} \frac{f(x) - f(g(y_0))}{x - g(y_0)}$, then gives

$$\lim_{n \longrightarrow \infty} \frac{f(g(y_n)) - f(g(y_0))}{g(y_n) - g(y_0)} = f'(g(y_0)) = f'(x_0) \neq 0. \tag{4.30}$$

Thus ends the proof. □

Example 4.13. Assume that we are given $\tan'(x) = \frac{1}{\cos^2 x}$, find $\arctan'$.
Solution. We have

$$\arctan'(y) = \frac{1}{\tan'(x)} = \cos^2(x). \tag{4.31}$$

What we need now is to represent $\cos^2(x)$ by $y = \tan x$. It is clear that $\cos^2 x = \frac{1}{1 + y^2}$ so $\arctan'(y) = \frac{1}{1 + y^2}$.

Example 4.14. Assume that we are given $(e^x)' = e^x$. Find $(\ln x)'$.
Solution. We have

$$(\ln)'(y) = \frac{1}{(e^x)'} = \frac{1}{e^x} = \frac{1}{y} \tag{4.32}$$

since $y = e^x$.

Example 4.15. **($f'(x_0) = 0$)** Consider $f(x) = x^3$. Then $g(y) = y^{1/3}$. We see that at $x_0 = 0$, g is not differentiable.

Theorem 4.16. (A Toy L'Hospital Rule) *Let f, g be differentiable at x_0, and furthermore $f(x_0) = g(x_0) = 0$. Then if $g'(x_0) \neq 0$, we have*

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{4.33}$$

Proof. We have

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow x_0} \frac{(f(x) - f(x_0))/(x - x_0)}{(g(x) - g(x_0))/(x - x_0)} = \frac{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}} = \frac{f'(x_0)}{g'(x_0)} \tag{4.34}$$

Thus ends the proof. □

Example 4.17. We have

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{(x^2 - 1)'|_{x=1}}{(x - 1)'|_{x=1}} = \frac{2}{1} = 2. \tag{4.35}$$

$$\lim_{x \longrightarrow 0} \frac{\sin x}{x} = \frac{\cos 0}{1} = 1. \tag{4.36}$$

Remark 4.18. The applicability of the above Toy L'Hospital rule is limited. For example, it cannot deal with $\lim_{x \longrightarrow 0} \frac{1 - \sin x}{x^2}$. We need the real L'Hospital rule for that.

4.2. Mean Value Theorem.

4.2.1. The Theorem.

Definition 4.19. (Local maximum/minimum) *Let $f: [a, b] \mapsto \mathbb{R}$ be a real function. We say f has a local maximum at $x_0 \in (a, b)$ if there exists some $\delta > 0$ such that $f(x) \leqslant f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. This x_0 is said to be a local maximizer. We say f has a local minimum at x_0 if there exists some $\delta > 0$ such that $f(x) \geqslant f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. This x_0 is said to be a local minimizer.*

Example 4.20. Let $f(x) = 1$ for all $x \in \mathbb{R}$ be the constant function. Then every $x \in \mathbb{R}$ is both a local maximizer and a local minimizer.

Example 4.21. Consider $f(x) = \sin(1/x)$ defined over $x \neq 0$. Then its local maximums are $\frac{2}{(4k+1)\pi}$, $k \in \mathbb{Z}$ while its local minimums are $\frac{2}{(4k+3)\pi}$, $k \in \mathbb{Z}$.

Theorem 4.22. *If f is differentiable at its local maximizer or minimizer, then the derivative is 0 there.*

Proof. Assume x_0 is a local maximizer. Take $x_n \in (x_0, x_0 + \delta)$ with $\lim_{n \longrightarrow \infty} x_n = x_0$. Since f is differentiable at x_0, we have

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{n \longrightarrow \infty} \frac{f(x_n) - f(x_0)}{x - x_0}. \tag{4.37}$$

But as $f(x_n) - f(x_0) \leqslant 0$ for all n, by comparison theorem we reach $f'(x_0) \leqslant 0$.

Now take $x_n \in (x_0 - \delta, x_0)$ with $\lim_{n \rightarrow \infty} x_n = x_0$. Similar argument as above gives $f'(x_0) \geqslant 0$. Therefore $f'(x_0) = 0$.

The proof for the local minimizer case is similar and left as exercise. □

Remark 4.23. It may happen that f is not differentiable at its maximizer or minimizer. For example $f(x) = |x|$.

Remark 4.24. Theorem 4.22 may be the most useful analysis theorem in real life, where the need for finding maximizer/minimizer of certain functions (representing cost, profit, ...) is ever increasing.

Example 4.25. Consider $f(x) = x \sin(1/x)$. Then its local maximizers and minimizers can be obtained by solving

$$0 = f'(x) = \sin(1/x) - \frac{x}{x^2} \cos(1/x) \Longrightarrow \tan(1/x) = 1/x. \tag{4.38}$$

The solutions have to be obtained numerically as it is not possible to represent them using elementary functions.

Theorem 4.26. (Rolle's Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there is $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Remark 4.27. Before proving the theorem, we illustrate the necessity of the assumptions.

- f is continuous on $[a, b]$. If not, $f(x) = \begin{cases} x & 0 \leqslant x < 1 \\ 0 & x = 1 \end{cases}$.
- f is differentiable on (a, b). If not, $f(x) = |x|$ over $[-1, 1]$.

Proof. Since f is continuous on $[a, b]$, there are $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min})$ is the minimum and $f(x_{\max})$ is the maximum. If one of them is different from a, b, then $f' = 0$ there due to Theorem 4.22. Otherwise we have $f(a) = f(b) = f(x_{\min}) = f(x_{\max}) \Longrightarrow f(x)$ is constant on $[a, b]$, consequently $f'(x) = 0$ for all $x \in (a, b)$. □

Theorem 4.28. **(Rolle over $\mathbb{R}$)** *Let f be continuous and differentiable on $\mathbb{R}$. If $\lim_{x \longrightarrow +\infty} f(x) = \lim_{x \longrightarrow -\infty} f(x)$, then there is $\xi \in \mathbb{R}$ such that $f'(\xi) = 0$.*

Proof. We discuss three cases. Let $\lim_{x \longrightarrow \infty} f = \lim_{x \longrightarrow -\infty} f = a$. Consider $A := \sup_{\mathbb{R}} f$ and $B := \inf_{\mathbb{R}} f$. Since f is continuous on $\mathbb{R}$, $A, B \in \mathbb{R}$ cannot be infinity.[4.1]

If $A = B = a$, then f is constant and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Otherwise, we have either $A > a$ or $B < a$ (or both). Assume $A > a$ (the case $B < a$ is similar). Take x_n such that $f(x_n) \longrightarrow \sup_{\mathbb{R}} f > a$. Since $\lim_{|x| \longrightarrow \infty} f(x) = a$, there is $M > 0$ such that $f(x) < (\sup_{\mathbb{R}} f + a)/2$ for all $|x| > M$. Consequently there is $N \in \mathbb{N}$ such that for all $n > N$, $|x_n| \leqslant M$. Now apply Bolzano-Weierstrass theorem, we have a subsequence $x_{n_k} \longrightarrow x_0 \in \mathbb{R}$. By continuity of f we have

$$f(x_0) = \lim_{k \longrightarrow \infty} f(x_{n_k}) = \sup_{\mathbb{R}} f(x) \tag{4.39}$$

therefore x_0 is a maximizer of f. Consequently $f'(x_0) = 0$. □

Theorem 4.29. **(Mean Value Theorem)** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{4.40}$$

Proof. Set $g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$ and apply Rolle's Theorem. □

Remark 4.30. When the interval has infinite size, the Mean Value Theorem may not hold (even if we accept $(f(b) - f(a))/\infty = 0$). An example is $f(x) = \arctan x$.

4.2.2. Applications.

Theorem 4.31. *Let f be defined over $[a, b] \subseteq \mathbb{R}$. Here a, b can be extended real numbers. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then*

a) *f is increasing if and only if $f'(x) \geqslant 0$ for all $x \in (a, b)$; f is decreasing if and only if $f'(x) \leqslant 0$ for all $x \in (a, b)$.*

b) *f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$; f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$.*

c) *f is a constant if and only if $f'(x) = 0$ for all $x \in (a, b)$.*

Proof.

a) We prove the increasing case here.

Let f be increasing, we show $f'(x) \geqslant 0$. Take any $x_0 \in (a, b)$. Since f is increasing, $f(x) \geqslant f(x_0)$ when $x > x_0$ and $f(x) \leqslant f(x_0)$ when $x < x_0$, thus

$$\frac{f(x) - f(x_0)}{x - x_0} \geqslant 0 \tag{4.41}$$

for all $x \neq x_0$. As f is differentiable at x_0, taking limit of both sides leads to $f'(x_0) \geqslant 0$.

Let $f'(x) \geqslant 0$ for all $x \in (a, b)$. Assume f is not increasing. Then there are $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Apply Mean Value Theorem we have there must exist $\xi \in (x_1, x_2) \subseteq (a, b)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} < 0. \tag{4.42}$$

4.1. If $\lim_{n \to \infty} f(x_n) = \infty$, then there is $N \in \mathbb{N}$ such that for all $n > N$, $f(x_n) > a + 1$. On the other hand, since $\lim_{x \to \infty} f = \lim_{x \to -\infty} f(x) = a$, there are M_1, M_2 such that $|f(x) - a| < 1$ when $x > M_1$ or $x < M_2$. Consequently $x_n \in [M_2, M_1]$ for all $n > N$. since $[M_2, M_1]$ is a bounded interval, there is subsequence $x_{n_k} \longrightarrow \xi \in [M_2, M_1]$. As a consequence of f's continuity we have $f(\xi) = \infty$. Contradiction.

Contradiction.

b) The proof is similar to the corresponding part of a).

c) The proof is left as exercise. □

Remark 4.32. Note that $f(x)$ strictly increasing $\not\Longrightarrow$ $f'(x) > 0$ everywhere. An examples is $f(x) = x^3$.

Example 4.33. Prove that $e^x > 1 + x$ for all $x > 0$.

Proof. Let $f(x) = e^x - 1 - x$. We see that $f(0) = 0$. To show $f(x) > 0$ it suffices to show f is strictly increasing. Calculate

$$f'(x) = e^x - 1 > 0 \tag{4.43}$$

for all $x > 0$. Therefore f is strictly increasing and consequently $f(x) > 0$ for all $x > 0$. □

Example 4.34. Prove

$$\frac{x}{1+x} \leqslant \ln(1+x) \leqslant x \tag{4.44}$$

for all $x > -1$.

Proof. For the first inequality let $f(x) = \ln(1+x) - \frac{x}{1+x}$. We have $f(0) = 0$ so all we need to show is $f(x) \geqslant f(0)$. Calculate

$$f'(x) = \frac{x}{(1+x)^2}. \tag{4.45}$$

Thus $f(x) \geqslant 0$ when $x > 0$ and $f(x) \leqslant 0$ when $x < 0$. Consequently $f(x) \geqslant f(0)$.

For the second inequality let $g(x) = x - \ln(1+x)$. We have $g(0) = 0$ and need to show $g(x) \geqslant g(0)$ for all x. Calculate

$$g'(x) = \frac{x}{1+x}. \tag{4.46}$$

For $x > -1$ we have $g'(x) > 0$ if $x > 0$ and <0 if $x < 0$. □

Example 4.35. Prove

$$\arctan\frac{1+x}{1-x} = \arctan x + \frac{\pi}{4} \tag{4.47}$$

for $-1 < x < 1$.

Proof. Set $x = 0$ we have

$$\arctan\frac{1+0}{1-0} = \arctan 0 + \frac{\pi}{4}. \tag{4.48}$$

Therefore all we need to show is

$$h(x) := \arctan\frac{1+x}{1-x} - \arctan x \tag{4.49}$$

is a constant for $-1 < x < 1$. Once this is shown, we have $h(x) = h(0) = \frac{\pi}{4}$.

Taking derivative, we have

$$h'(x) = \frac{\left(\frac{1+x}{1-x}\right)'}{1 + \left(\frac{1+x}{1-x}\right)^2} - \frac{1}{1+x^2} = \frac{\frac{1\cdot(1-x) - (-1)\cdot(1+x)}{(1-x)^2}}{\frac{(1-x)^2 + (1+x)^2}{(1-x)^2}} - \frac{1}{1+x^2} = 0. \tag{4.50}$$

Thus ends the proof. □

4.2.3. L'Hospital's Rule.

We have seen that if f, g are differentiable at x_0 and $g'(x_0) \neq 0$, then

$$\lim \frac{f}{g} = \frac{f'(x_0)}{g'(x_0)}. \tag{4.51}$$

More generally, we have

Theorem 4.36. (L'Hospital's Rule) *Let $x_0 \in (a,b)$ and $f(x), g(x)$ be differentiable on $(a,b) \backslash \{x_0\}$. Assume that $\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = 0$. Then if $\lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ for $x \in (a,b)$, the following holds.*

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)}. \tag{4.52}$$

To prove it, we need the following

Theorem 4.37. (Cauchy's extended mean value theorem) *Let f, g be continuous over $[a, b]$ and differentiable over (a,b). Then there is $\xi \in (a,b)$ such that*

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}. \tag{4.53}$$

Proof. Take

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x) \tag{4.54}$$

we have $h(a) - h(b) = 0 \Longrightarrow h(a) = h(b)$. Application of the mean value theorem gives the desired result. □

Proof. (of L'Hospital's Rule) Since $\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = 0$ we can define $f(x_0) = g(x_0) = 0$. After such definition f, g becomes continuous over (a,b). Now for any $x \in (a,b)$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} \tag{4.55}$$

for some ξ between x, x_0, thanks to the extended mean value theorem. Now taking limit $x \longrightarrow x_0$, we have $\xi \longrightarrow x_0$ and the conclusion follows. □

Example 4.38. Find $\lim_{x \longrightarrow 0} \frac{x \sin x}{x^2}$.

We see that the conditions for L'Hospital's rule is satisfied. Therefore

$$\lim_{x \longrightarrow 0} \frac{x \sin x}{x^2} = \lim_{x \longrightarrow 0} \frac{\sin x + x \cos x}{2\,x} = \lim_{x \longrightarrow 0} \frac{2 \cos x - x \sin x}{2} = 1. \tag{4.56}$$

Remark 4.39. L'Hospital's rule still holds when $x_0 = \pm\infty$, $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \pm\infty$, or $\lim_{x \longrightarrow x_0} f, \lim_{x \longrightarrow x_0} g = \pm\infty$. The proofs for these generalizations are not required.

Example 4.40. Find $\lim_{x \longrightarrow 0} x \ln x$. We have

$$\lim_{x \longrightarrow 0} x \ln x = \lim_{x \longrightarrow 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \longrightarrow 0} (-x) = 0. \tag{4.57}$$

Remark 4.41. L'Hospital's rule only applies to the situations $0/0$, $(\pm\infty)/(\pm\infty)$.

4.3. Taylor Expansion.

4.3.1. Derivative from approximation point of view.

Recall the definition of derivative:

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0). \tag{4.58}$$

We can re-write it as

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0) - f'(x_0)\,(x - x_0)}{x - x_0} = 0. \tag{4.59}$$

Now consider the following problem: Given $f(x)$ differentiable at x_0. Find the best first order polynomial $g(x) = a + b\,(x - x_0)$ to approximate $f(x)$.

Theorem 4.42. *The function $G(x) := f(x_0) + f'(x_0)\,(x - x_0)$ is the best first order polynomial approximate of f at x_0, in the following sense: Let $g(x) = a + b\,(x - x_0)$ be any other first order polynomial, then*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - G(x)}{f(x) - g(x)} = 0 \tag{4.60}$$

Proof. First if $a \neq f(x_0)$, we have

$$\lim_{x \longrightarrow x_0} [f(x) - G(x)] = 0, \qquad \lim_{x \longrightarrow x_0} [f(x) - g(x)] = f(x_0) - a \neq 0 \tag{4.61}$$

so (4.60) holds.

Now consider $g(x) = f(x_0) + b\,(x - x_0)$. We have

$$\lim_{x \longrightarrow x_0} f(x) - G(x) = \lim_{x \longrightarrow x_0} f(x) - g(x) = 0 \tag{4.62}$$

therefore can apply L'Hospital's rule to reach

$$\lim_{x \longrightarrow x_0} \frac{f(x) - G(x)}{f(x) - g(x)} = \lim_{x \longrightarrow x_0} \frac{f'(x) - f'(x_0)}{f'(x) - b} \tag{4.63}$$

which equals 0 unless $b = f'(x_0)$. □

Remark 4.43. Note that Theorem 4.42 can also be proved directly, without using L'Hospital's rule:

$$\frac{f(x) - G(x)}{f(x) - g(x)} = \frac{f(x) - f(x_0) - f'(x_0)\,(x - x_0)}{f(x) - f(x_0) - b\,(x - x_0)} = \frac{\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)}{\frac{f(x) - f(x_0)}{x - x_0} - b} \tag{4.64}$$

Taking limit gives the desired result.

4.3.2. Higher order derivatives.

Let $f(x)$ be differentiable on $(a,\, b)$. Then $f'(x)$ is defined for all $(a,\, b)$ and we can talk about its differentiability and define second order derivative $f''(x)$. Similarly we can define $f'''(x)$ and higher order derivatives. In short, we define $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$.

Example 4.44. Let $f(x) = e^{3x}$. Compute $f^{(3)}(x)$.

We have

$$f^{(3)}(x) = ((f')')' = ((3\,e^{3x})')' = (9\,e^{3x})' = 27\,e^{3x}. \tag{4.65}$$

Note that for $f^{(n)}(x_0)$ to exist, $f^{(n-1)}(x)$ must exist over $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

4.3.3. Taylor expansion.

Theorem 4.45. *Let f be such that $f^{(k)}(x)$ exists on (a, b) for $k = 1, 2, \ldots, n-1$, and $f^{(n)}(x_0)$ exists for $x_0 \in (a, b)$. Denote $P_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n$. Then $P_n(x)$ is the best approximate polynomial for f at x_0 in the sense that for any other polynomial $Q_n(x)$ of order n, we have*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - P_n(x)}{f(x) - Q_n(x)} = 0 \tag{4.66}$$

Proof. Let $Q_n(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n$. First observe that if $q_0 \neq f(x_0)$, then

$$\lim_{x \longrightarrow x_0} \frac{f(x) - P_n(x)}{f(x) - Q_n(x)} = \frac{0}{f(x_0) - q_0} = 0. \tag{4.67}$$

If $q_0 = f(x_0)$ but $q_1 \neq f'(x_0)$, we have a $\frac{0}{0}$ type ratio and can apply L'Hospital's rule:

$$\lim_{x \longrightarrow x_0} \frac{f(x) - P_n(x)}{f(x) - Q_n(x)} = \lim_{x \to x_0} \frac{f'(x) - [f'(x_0) + f''(x_0)(x - x_0) + \cdots]}{f'(x) - [q_1 + q_2(x - x_0) + \cdots]} = \frac{0}{f'(x_0) - q_1} = 0. \tag{4.68}$$

Doing this repeatedly, we see that the limit is 0 unless $Q_n = P_n$. □

If f has better differentiability, we can write $f(x) - P_n(x)$ out more explicitly.

Theorem 4.46. (Lagrange form of the remainder) *Let f be such that $f^{(k)}(x)$ exists on (a, b). Then for every $x, x_0 \in (a, b)$ the following holds:*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \tag{4.69}$$

where ξ is between x, x_0.

Remark 4.47. It is important to understand that ξ depends on x, that is when x changes, so does ξ. For any fixed x, it is clear that there is $r \in \mathbb{R}$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r(x - x_0)^{n+1}. \tag{4.70}$$

Thus what the theorem actually says is: $\exists \xi$ between x, x_0 such that $r = \frac{f^{(n+1)}(\xi)}{(n+1)!}$.

Proof. In the following x is fixed. Take $r \in \mathbb{R}$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r(x - x_0)^{n+1}. \tag{4.71}$$

holds for **this particular** x.

Now set

$$g(t) = f(t) - \left[f(x_0) + f'(x_0)(t - x_0) + \frac{f''(x_0)}{2}(t - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n + r(t - x_0)^{n+1} \right] \tag{4.72}$$

then we have $g(x_0) = g(x) = 0$. Applying Rolle's theorem, we obtain ξ_1 between x_0, x such that $g'(\xi_1) = 0$. On the other hand clearly $g'(x_0) = 0$. Thus we have ξ_2 between ξ_1 and x_0 (thus also between x, x_0) such that $g''(\xi_2) = 0$. Apply this n times we conclude that there is ξ such that $g^{(n+1)}(\xi) = 0$, which gives

$$r = \frac{f^{(n+1)}(\xi)}{(n+1)!}. \tag{4.73}$$

□

Remark 4.48. Note that the case $n = 0$ is exactly Rolle's theorem. Also note that one cannot prove the above theorem through induction.

Definition 4.49. (Taylor Polynomial) *The polynomial*

$$f(x_0) + f'(x_0)(t - x_0) + \frac{f''(x_0)}{2}(t - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n \tag{4.74}$$

is called the Taylor polynomial of the function f, the term

$$R_n := \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \tag{4.75}$$

is called the Lagrange form of the remainder.

Example 4.50. Calculate Taylor polynomial with Lagrange form of remainder (to degree 2 – that is $n=2$) of the following functions at $x_0=0$.

$$f(x)=\sin(\sin x); \qquad f(x)=x^4+x+1; \qquad f(x)=\frac{1}{1+x^2} \tag{4.76}$$

Solution.

- $f(x)=\sin(\sin x)$. We calculate:

$$f'(x)=[\cos(\sin x)]\cos x \Longrightarrow f'(0)=1; \tag{4.77}$$

$$f''(x)=[-\sin(\sin x)\cos x]\cos x-[\cos(\sin x)]\sin x \Longrightarrow f''(0)=0; \tag{4.78}$$

$$\begin{aligned} f'''(x) &= \{[-\sin(\sin x)]\cos^2 x\}'-\{[\cos(\sin x)]\sin x\}' \\ &= -\cos(\sin x)\cos^3 x+2\sin(\sin x)\cos x\sin x \\ &\quad +\sin(\sin x)\sin x\cos x-\cos(\sin x)\cos x \\ &= -\cos x\,[(\cos^2 x+1)\cos(\sin x)-3\sin x\,(\sin(\sin x))]. \end{aligned} \tag{4.79}$$

  Thus the Taylor polynomial at $x_0=0$ to degree 2 reads:

$$0+1\cdot(x-0)+\frac{0}{2}(x-0)^2+\frac{f'''(\xi)}{6}(x-0)^3 \tag{4.80}$$

  which simplifies to

$$\sin(\sin x)=x+\frac{-\cos\xi\,[(\cos^2\xi+1)\cos(\sin\xi)-3\sin\xi\,(\sin(\sin\xi))]}{6}x^3. \tag{4.81}$$

  Here ξ lies between 0 and x.

- $f(x)=x^4+x+1$. We calculate:

$$f(0)=1,\;\; f'(x)=4x^3+1 \Longrightarrow f'(0)=1,\;\; f''(x)=12x^2 \Longrightarrow f''(0)=0 \tag{4.82}$$

  and

$$f'''(x)=24x. \tag{4.83}$$

  Therefore the Taylor polynomial at $x_0=0$ to degree 2 reads

$$x^4+x+1=1+x+(4\xi)x^3 \tag{4.84}$$

  where ξ lies between 0 and x.

- $f(x)=\frac{1}{1+x^2}$. We calculate:

$$f(0)=1; \qquad f'(x)=-\frac{2x}{(1+x^2)^2} \Longrightarrow f'(0)=0 \tag{4.85}$$

$$f''(x)=-\frac{2(1+x^2)^2-8x^2(1+x^2)}{(1+x^2)^4}=\frac{6x^2-2}{(1+x^2)^3} \Longrightarrow f''(0)=-2. \tag{4.86}$$

$$f'''(x)=\frac{12x(1+x^2)^3-6x(1+x^2)^2(6x^2-2)}{(1+x^2)^6}=\frac{24x(1-x^2)}{(1+x^2)^4}. \tag{4.87}$$

  Therefore the Taylor polynomial at $x_0=0$ to degree 2 reads

$$\frac{1}{1+x^2}=1-x^2+\frac{4\xi(1-\xi^2)}{(1+\xi^2)^4}x^3 \tag{4.88}$$

  where ξ lies between 0 and x.

Example 4.51. Calculate Taylor polynomial (to degree 2) of the following functions at the specified x_0's.

$$f(x)=\sin x,\;\; x_0=\frac{\pi}{2}; \qquad f(x)=x^4+x+1,\;\; x_0=1; \qquad f(x)=e^x,\;\; x_0=2. \tag{4.89}$$

Solution.

- $f(x)=\sin x, x_0=\frac{\pi}{2}$.

We have

$$f(x_0) = \sin\left(\frac{\pi}{2}\right) = 1; \tag{4.90}$$

$$f'(x) = \cos x \Longrightarrow f'(x_0) = 0; \tag{4.91}$$

$$f''(x) = -\sin x \Longrightarrow f''(x_0) = -1; \tag{4.92}$$

$$f'''(x) = -\cos x. \tag{4.93}$$

Therefore the answer is

$$\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{\cos \xi}{6}\left(x - \frac{\pi}{2}\right)^3 \tag{4.94}$$

where ξ is between x and $\pi/2$.

- $f(x) = x^4 + x + 1,\ x_0 = 1.$

  We have

$$f(x_0) = 3; \qquad f'(x) = 4\,x^3 + 1 \Longrightarrow f'(x_0) = 5 \tag{4.95}$$

$$f''(x) = 12\,x^2 \Longrightarrow f''(x_0) = 12; \qquad f'''(x) = 24\,x. \tag{4.96}$$

So the answer is

$$x^4 + x + 1 = 3 + 5\,(x-1) + 6\,(x-1)^2 + 4\,\xi\,(x-1)^3 \tag{4.97}$$

where ξ is between x and 1.

- $f(x) = e^x,\ x_0 = 2.$

  We have

$$f(x_0) = f'(x_0) = f''(x_0) = e^2, \qquad f'''(x) = e^x. \tag{4.98}$$

So the answer is

$$e^x = e^2 + e^2\,(x-2) + \frac{e^2}{2}\,(x-2)^2 + \frac{e^\xi}{6}\,(x-2)^3 \tag{4.99}$$

where ξ is between x and 1.

Remark 4.52. Note that the Taylor polynomial is just the best approximation at x_0. Therefore are naturally different when x_0 changes.

Example 4.53. Prove the following.

a) $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ for all $x > 0$.

b) $\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| < \frac{1}{24}$ for all $x \in (-1, 1)$.

Proof.

a) The Taylor polynomial with Lagrange remainder for e^x at 0 is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{e^\xi}{24}\,x^4. \tag{4.100}$$

Since $x > 0$, ξ (note that it depends on x, that is $\xi = \xi(x)$ is in fact a function of x) is also positive. Consequently $\frac{e^\xi}{24}\,x^4 > 0$ for all x. So $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ holds for all $x > 0$.

b) The Taylor polynomial with Lagrange remainder for $\cos x$ at 0 is (up to degree 2):

$$\cos x = 1 - \frac{x^2}{2} + \frac{\sin \xi}{6}\,x^3 \tag{4.101}$$

with ξ between 0 and x. Thus we have

$$\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| = \frac{|\sin \xi|}{6}\,|x|^3 < \frac{1}{6}. \tag{4.102}$$

for all $x \in (-1,1)$. This is not enough so we expand one more term:

$$\cos x = 1 - \frac{x^2}{2} + \frac{\cos \xi}{24} x^4 \tag{4.103}$$

which gives

$$\left| \cos x - \left(1 - \frac{x^2}{2}\right) \right| = \frac{|\cos \xi|}{24} |x|^4 < \frac{1}{24}. \tag{4.104}$$

□

Definition 4.54. (Taylor series) *If $f^{(n)}(x)$ exists for all $n \in \mathbb{N}$ over (a,b), then for any $x_0 \in (a,b)$ one can write down an infinite series (polynomial of infinite degree):*

$$f(x_0) + f'(x_0)(t - x_0) + \frac{f''(x_0)}{2}(t - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{4.105}$$

This is called the Taylor series of f at x_0.

Remark 4.55. Note that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ may not hold. A counterexample is $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ whose Taylor series is $\sum_{n=0}^{\infty} 0 \cdot (x - x_0)^n$.

Example 4.56. We can calculate the Taylor series e^x, $\cos x$.

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}; \qquad \cos x \sim \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{4.106}$$