## 2. Sets in $\mathbb{R}$

### 2.1. Sup and Inf.

#### 2.1.1. Definitions.

**Definition 2.1. (sup and inf)** *Let $A$ be a nonempty set of numbers. The supreme of $A$ is defined as*

$$\sup A = \min \{b \in \mathbb{R}: b \geqslant a \text{ for every } a \in A\}. \tag{2.1}$$

*If $\{b \in \mathbb{R}: b \geqslant a \text{ for every } a \in A\} = \varnothing$, we write*

$$\sup A = \infty; \tag{2.2}$$

*The infimum of $A$ is defined as*

$$\inf A = \max \{b \in \mathbb{R}: b \leqslant a \text{ for every } a \in A\}. \tag{2.3}$$

*If $\{b \in \mathbb{R}: b \geqslant a \text{ for every } a \in A\} = \varnothing$ , write*

$$\inf A = -\infty. \tag{2.4}$$

sup and inf are generalizations of max and min.

**Proposition 2.2.** $\sup A = \max A, \inf A = \min A$ *when* $\max A, \min A$ *exist. In particular, when $A$ is a finite set (that is has finitely many elements), it always holds that* $\sup A = \max A$*;* $\inf A = \min A$.

**Proof.** Let $a_{\max} = \max A$. Set $B = \{b \in \mathbb{R}: b \geqslant a \text{ for every } a \in A\}$. We need to show that $a_1 = \min B$, that is

1. $a_{\max} \in B$. As $a_{\max} = \max A$, we have $a_{\max} \geqslant a$ for all $a \in A$. Therefore $a_1 \in B$;
2. $\forall b \in B$, $a_{\max} \leqslant b$. Take any $b \in B$. Then $b \geqslant a$ for all $a \in A$. In particular $b \geqslant a_{\max}$.

The proof for the inf/min part is similar. □

**Example 2.3. (max/min may not exist)** Let $A = \left\{1 - \frac{1}{n}: n \in \mathbb{N}\right\}$. Then $\sup A = 1$, $\inf A = \min A = 0$, while $\max A$ does not exist.

- $\sup A = 1$. We show two things:
  1. $\forall a \in A, 1 \geqslant a$. Take any $a \in A$. Then there is $n \in \mathbb{N}$ such that $a = 1 - \frac{1}{n} < 1$.
  2. $\forall b \in \mathbb{R}$ such that $b \geqslant a$ for all $a \in A$, $b \geqslant 1$. Since $b \geqslant a$ for all $a \in A$, $b \geqslant 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Assume $b < 1$. Taking $n > \frac{1}{1 - b}$ leads to contradiction.
- $\inf A, \min A = 0$. Omitted.
- $\max A$ does not exist. Assume the contrary, then there is $a_{\max} \in A$. Then there is $n_0 \in \mathbb{N}$ such that $a_{\max} = 1 - \frac{1}{n_0}$. Taking $n > n_0$ we have $a_{\max} < 1 - \frac{1}{n} \in A$, contradiction.

Although max/min may not exist, sup/inf always does.

**Theorem 2.4.** *Let $A$ be a nonempty set of numbers, then* $\sup A, \inf A$ *exist.*

**Proof.** The existence of $\sup A$ follows directly from the least upper bound property of $\mathbb{R}$. To show that $\inf A$ exists, consider the set $B := \{b \in \mathbb{R}: b \leqslant a \text{ for every } a \in A\}$. Since $A$ is nonempty, there is at least one $a \in A$. By definition of $B$ this $a$ is an upper bound of $B$. Thus the least upper bound property leads to the existence of $\sup B \in \mathbb{R}$. All we need to show is that $b_{\max} := \sup B \in B$.

Assume the contrary, that is $b_{\max} \notin B$. Then there is $a \in A$ such that $b_{\max} > a$. Let $\tilde{b} := \frac{b_{\max} + a}{2}$. Then we have $\tilde{b} > a$ so for every $b \in B$, $\tilde{b} > b$. This contradicts the fact that $b_{\max} = \sup B = \min \left\{\tilde{b} \in \mathbb{R}: \tilde{b} \geqslant b \text{ for every } b \in B\right\}$. □

**Theorem 2.5. (Approximation of sup and inf)** *Let $A \subseteq \mathbb{R}$ with $\sup A, \inf A \in \mathbb{R}$. Then for every $\varepsilon > 0$, there are $a, b \in A$ such that*

$$\sup A - a < \varepsilon; \qquad b - \inf A < \varepsilon \tag{2.5}$$

**Proof.** We prove the sup case and left the inf case as exercise. Assume the contrary. Then there is $\varepsilon_0 > 0$ such that for all $a \in A$, $\sup A - a \geqslant \varepsilon_0$. Now set $a_{\sup} := \sup A - \varepsilon_0/2$. We have $a_{\sup} > a$ for all $a \in A$ but $a_{\sup} < \sup A$. Contradiction. □

**Remark 2.6.** By setting $\varepsilon = 1/n$, we can obtain a sequence $\{x_n\}$ with $x_n \in A$ such that $x_n \longrightarrow \sup A$ (or $\inf A$). However note that $x_n$ may not be different elements from one another. For example when $A$ is finite, we basically will have to take the sequence $x_n = a_{\max}$.

### 2.1.2. Sup/Inf of functions.

A special case that is particularly useful is when the sets are images of functions.

**Definition 2.7. (Sup and Inf of functions)** *Let $f: E \mapsto \mathbb{R}$ be a function. We define*

$$\sup_{x \in E} f(x) := \sup f(E); \qquad \inf_{x \in E} f(x) := \inf f(E). \tag{2.6}$$

**Example 2.8.** $\sup_{x \in (-1,1)} x^2 = 1$, $\inf_{x \in (-1,1)} x^2 = 0$.

**Theorem 2.9. (sup and inf under operations)** *Let $f, g$ be functions with domains containing $E \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$ be a* positive *number. Then*

*a)* $\sup_{x \in E} (c f) = c \sup_{x \in E} f$; $\inf_{x \in E} (c f) = c \inf_{x \in E} f$;

*b)* $\sup_{x \in E} (-f) = -\inf_{x \in E} f$; $\inf_{x \in E} (-f) = -\sup_{x \in E} f$;

*c)* $\sup_{x \in E} (f + g) \leqslant \sup_{x \in E} f + \sup_{x \in E} g$;

*d)* $\inf_{x \in E} (f + g) \geqslant \inf_{x \in E} f + \inf_{x \in E} g$.

*The above holds even when the supreme/infimum is $\infty$ or $-\infty$.*

**Proof.** We only prove b), c) here. Other cases are left as exercise.

- b). We show $\sup_{x \in E} (-f) = -\inf_{x \in E} f$. Once this is done setting $g = -f$ gives the other half. Let $a = \inf_{x \in E} f$ (maybe $-\infty$). We need to show

  1. $-a \geqslant -f(x)$ for all $x \in E$. Since $a = \inf_{x \in E} f$, by definition $a \leqslant f(x)$ for all $x \in E$. Therefore $-a \geqslant -f(x)$ for all $x \in E$.

  2. For any $b \in \mathbb{R}$ satisfying $b \geqslant -f(x)$ for all $x \in E$, we have $b \geqslant -a$. Since $b \geqslant -f(x)$ for all $x \in E$, we have $-b \leqslant f(x)$ for all $x \in E$. Since $a = \inf_{x \in E} f$, $a \geqslant -b$. Therefore $b \geqslant -a$.

Note that the above argument still holds when $a = -\infty$.

- c). Denote $a = \sup_{x \in E} f$, $b = \sup_{x \in E} g$, we need to show that $a + b \geqslant f(x) + g(x)$ for all $x \in E$. If one of $a, b$ is $\infty$, then we have $a + b = \infty \geqslant f(x) + g(x)$ for all $x \in E$.[2.1] If both $a, b \in \mathbb{R}$, take any $x \in E$. We have $a = \sup_{x \in E} f \geqslant f(x)$ and $b = \sup_{x \in E} g \geqslant g(x)$. Consequently $a + b \geqslant f(x) + g(x)$. □

**Example 2.10.** The inequalities in c),d) may hold strictly. The reason is that the "peak" of $f$ and the "peak" of $g$ may not be at the same location. For example, take $f = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ and $g = \begin{cases} 1 & 1 \leqslant x < 2 \\ 0 & \text{elsewhere} \end{cases}$, we have $\sup_{x \in \mathbb{R}} f = \sup_{x \in \mathbb{R}} g = 1$, but $\sup_{x \in \mathbb{R}} (f + g) = 1 < 1 + 1 = 2$.

**Theorem 2.11. (Monotone property)** *Suppose $A \subseteq B$ are nonempty subsets of $\mathbb{R}$. Then*

*a)* $\sup B \geqslant \sup A$.

*b)* $\inf B \leqslant \inf A$.

**Proof.** We prove a) and leave b) as exercise.

If $\sup B = \infty$, then $\sup B \geqslant \sup A$ holds; If $\sup B \in \mathbb{R}$, then by definition we have

$$\sup B \geqslant b \text{ for every } b \in B \Longrightarrow \sup B \geqslant a \text{ for every } a \in A \tag{2.7}$$

because $A \subseteq B$. By definition of $\sup A$ we conclude that $\sup B \geqslant \sup A$. □

### 2.1.3. Limits supreme and infimum.

**Definition 2.12.** *Let $\{x_n\}$ be a real sequence. The* limit supreme *of $\{x_n\}$ is the extended real number*

$$\limsup_{n \longrightarrow \infty} x_n := \lim_{n \longrightarrow \infty} \left( \sup_{k \geqslant n} x_k \right), \tag{2.8}$$

*and the* limit infimum *of $\{x_n\}$ is the extended real number*

$$\liminf_{n \longrightarrow \infty} x_n := \lim_{n \longrightarrow \infty} \left( \inf_{k \geqslant n} x_k \right). \tag{2.9}$$

**Remark 2.13.** Since $\sup_{k \geqslant n} x_k$ is decreasing and $\inf_{k \geqslant n} x_k$ is increasing, limsup and liminf always exist (in contrast to limits).

**Remark 2.14.** Note that it is possible that $\limsup_{n \longrightarrow \infty} x_n = -\infty$ and $\liminf_{n \longrightarrow \infty} = \infty$.

**Lemma 2.15.** *Let $\{x_n\}$ be a real sequence. Then*

$$\limsup_{n \longrightarrow \infty} x_n \geqslant \liminf_{n \longrightarrow \infty} x_n \tag{2.10}$$

**Proof.** Follows from Comparison Theorem. Left as exercise. □

**Theorem 2.16. (Relation to limits and subsequences)** *Let $\{x_n\}$ be a real sequence. Then*

*a)* $x_n \longrightarrow a$ *if and only if* $\limsup_{n \longrightarrow \infty} x_n = \liminf_{n \longrightarrow \infty} x_n = a$.[2.2]

2.1. Note that by definition sup can only be real number or $\infty$, while inf can only be real number or $-\infty$.

b) *There are two subsequences converging to* $\limsup_{n\longrightarrow\infty} x_n$ *and* $\liminf_{n\longrightarrow\infty} x_n$*, respectively.*

c) *If* $\{x_{n_k}\}$ *is a convergent subsequence, then* $\liminf_{n\longrightarrow\infty} x_n \leqslant \lim_{k\longrightarrow\infty} x_{n_k} \leqslant \limsup_{n\longrightarrow\infty} x_n$.

**Proof.**

a) We only prove the case $a\in\mathbb{R}$ and left $a=\pm\infty$ as exercise.

- "if". Assume $\limsup_{n\longrightarrow\infty} x_n = \liminf_{n\longrightarrow\infty} x_n = a$. Then as

$$\sup_{k\geqslant n} x_k \geqslant x_n \geqslant \inf_{k\geqslant n} x_k, \tag{2.11}$$

  application of Squeeze Theorem gives the convergence of $x_n$ as well as $\lim_{n\longrightarrow\infty} x_n = a$.

- "only if". For any $\varepsilon>0$, since $x_n \longrightarrow a$, there is $N\in\mathbb{N}$ such that $|x_n - a| < \varepsilon$. This implies $\sup_{n>N} x_n < a+\varepsilon$ and $\inf_{n>N} x_n > a-\varepsilon$. Therefore when $n>N$,

$$a-\varepsilon < \inf_{k\geqslant n} x_n \leqslant \sup_{k\geqslant n} x_n < a+\varepsilon. \tag{2.12}$$

  Comparison Theorem now gives

$$a-\varepsilon \leqslant \liminf_{n\longrightarrow\infty} x_n \leqslant \limsup_{n\longrightarrow\infty} x_n \leqslant a+\varepsilon. \tag{2.13}$$

  As this holds for all $\varepsilon>0$, we must have $\limsup_{n\longrightarrow\infty} x_n = \liminf_{n\longrightarrow\infty} x_n = a$.

b) We only show the existence of $x_{n_k} \longrightarrow \limsup_{n\longrightarrow\infty} x_n$. Thanks to Theorem 2.5, there is $n_1\in\mathbb{N}$ such that

$$\sup_{k\geqslant 1} x_k - 1 \leqslant x_{n_1} \leqslant \sup_{k\geqslant 1} x_k; \tag{2.14}$$

Now apply Theorem 2.5 to $\{x_{n_1+1},\ldots\}$, we obtain $n_2\in\mathbb{N}$ such that

$$\sup_{k\geqslant n_2+1} x_k - \frac{1}{2} \leqslant x_{n_2} \leqslant \sup_{k\geqslant n_2+1} x_k. \tag{2.15}$$

This way we obtain a subsequence satisfying

$$\sup_{k\geqslant n_l+1} x_k - \frac{1}{l} \leqslant x_{n_l} \leqslant \sup_{k\geqslant n_l+1} x_k. \tag{2.16}$$

We take limit of both sides. As $\{\sup_{k\geqslant n_l+1} x_k\}$ is a subsequence of $\{\sup_{k\geqslant n} x_k\}$, it converges to the same limit $a$. Using the fact that $1/l\longrightarrow 0$ as $l\longrightarrow\infty$, we apply Squeeze theorem to conclude $x_{n_l}\longrightarrow a$.

c) Let $\{x_{n_k}\}$ be the subsequence. Then we have

$$\inf_{l\geqslant n_k} x_l \leqslant x_{n_k} \leqslant \sup_{l\geqslant n_k} x_l. \tag{2.17}$$

Taking limit of all three and applying Comparison Theorem we reach the conclusion.

□

2.2. *We do not say "*$\{x_n\}$ *converges" to avoid dealing with* $a=\pm\infty$ *separately.*

### 2.2. A bit topology.

#### 2.2.1. Intervals.

**Definition 2.17.** *Let $a, b$ be real numbers. A closed interval is a set of the form*

$$[a, b] := \{x \in \mathbb{R}: a \leqslant x \leqslant b\}, \qquad [a, \infty) := \{x \in \mathbb{R}: a \leqslant x\} \tag{2.18}$$

$$(-\infty, b] := \{x \in \mathbb{R}: x \leqslant b\}; \qquad (-\infty, \infty) := \mathbb{R}. \tag{2.19}$$

*An open interval is a set of the form*

$$(a, b) := \{x \in \mathbb{R}: a < x < b\}, \qquad (a, \infty) := \{x \in \mathbb{R}: a < x\} \tag{2.20}$$

$$(-\infty, b) := \{x \in \mathbb{R}: x < b\}; \qquad (-\infty, \infty) := \mathbb{R}. \tag{2.21}$$

*One can also define half-open, half-closed intervals:*

$$[a, b): = \{x \in \mathbb{R}: a \leqslant x < b\}, \qquad (a, b] := \{x \in \mathbb{R}: a < x \leqslant b\}. \tag{2.22}$$

**Remark 2.18.** Note that $\mathbb{R}$ is both an open interval and a closed interval.

**Example 2.19.** Write the following in interval notation:

a) $A := \{x \in \mathbb{R}: |x - 3| \leqslant 1\}$;

b) $B := \{x \in \mathbb{R}: |x - 3| > 5\}$.

c) $(1, 2)^c$.

**Solution.**

a) For $A$ we have $A = \{x \in \mathbb{R}: 2 \leqslant x \leqslant 4\}$ so $A = [2, 4]$;

b) $B = \{x \in \mathbb{R}: x > 8 \text{ or } x < -2\}$ so $B = (-\infty, -2) \cup (8, \infty)$.

c) $(1, 2) = \{x \in \mathbb{R}: 1 < x < 2\}$ so $(1, 2)^c = \{x \in \mathbb{R}: x \leqslant 1 \text{ or } x \geqslant 2\} = \{x \in \mathbb{R}: x \leqslant 1\} \cup \{x \in \mathbb{R}: x \geqslant 2\}$ which equals $(-\infty, 1] \cup [2, \infty)$.

**Lemma 2.20.** *The following holds:*

*a)* $(a, b) \subseteq (c, d) \Longleftrightarrow a \geqslant c, b \leqslant d$; $(a, b) \subseteq [c, d] \Longleftrightarrow a \geqslant c, b \leqslant d$; $[a, b] \subseteq [c, d] \Longleftrightarrow a \geqslant c, b \leqslant d$.

*b)* $[a, b] \subseteq (c, d) \Longleftrightarrow a > c, b < d$.

#### 2.2.2. Open sets and closed sets.

**Definition 2.21.** *A set $E \subseteq \mathbb{R}$ is* open *if for every $x \in E$, there is an open interval $(a, b) \subseteq E$ such that $x \in (a, b)$. A set $E \subseteq \mathbb{R}$ is* closed *if its complement $E^c := \mathbb{R} \backslash E$ is open.*

**Remark 2.22.** Traditionally, we say $\mathbb{R}$ and $\varnothing$ are both open and closed.

**Lemma 2.23.** *Open intervals are open, closed intervals are closed. Half-open, half-closed intervals are neither open nor closed.*

**Proof.** Let $(a, b)$ be an open interval ($a, b$ may be extended real numbers so this covers all four cases). For every $x \in (a, b)$, we have $x \in (a, b) \subseteq (a, b)$. Therefore $(a, b)$ is open.

Next let $a, b \in \mathbb{R}$ and $[a, b]$ be an closed interval. Then we have $[a, b]^c = (-\infty, a) \cup (b, \infty)$. Take any $x \in [a, b]^c$. Then there are two cases:

- $x \in (-\infty, a)$. Since $(-\infty, a)$ is an open interval, we have $x \in (-\infty, a) \subseteq [a, b]^c$.
- $x \in (b, \infty)$. Similarly, $(b, \infty)$ is an open interval so we have $x \in (b, \infty) \subseteq [a, b]^c$.

The other three cases are easier:

- $[a, \infty)$: We have $[a, \infty)^c = (-\infty, b)$ open;

- $(-\infty, b]$: We have $(-\infty, b]^c = (b, \infty)$ open;
- $(-\infty, \infty)$: We have $(-\infty, \infty)^c = \varnothing$ open.

Finally we show that $[a, b)$ and $(a, b]$ are neither open nor closed. For $[a, b)$, to see that it is not open, take $x = a \in [a, b)$. Then for every open interval $(c, d)$ containing $x = a$, we have $c < a$ and therefore $\frac{c+a}{2} \in (c, d)$ but $\frac{c+a}{2} \notin [a, b)$. To see that it is not closed, we consider $[a, b)^c = (-\infty, a) \cup [b, \infty)$. Take $x = b$ and argue similarly, we see that $[a, b)^c$ is not open. Therefore $[a, b)$ is neither open nor closed. The proof for $(a, b]$ is similar. □

**Lemma 2.24.** *If $E$ is open, then $E^c$ is closed; If $E$ is closed then $E^c$ is open.*

**Proof.** The second part is by definion. For the first part, because $(E^c)^c = E$, $E^c$ is closed if $E$ is open. □

**Theorem 2.25.** *We have the following results about intersection and union of sets:*

*a) The intersection of finitely many open sets is open; The union of open sets is open.*

*b) The intersection of closed sets is closed; The union of finitely many closed sets is closed.*

**Proof.** We prove a). b) follows from a), Lemma 2.24, and De Morgan's rule of set operations:

$$(\cap_{\alpha \in A} E_\alpha)^c = \cup_{\alpha \in A} E_\alpha^c; \qquad (\cup_{\alpha \in A} E_\alpha)^c = \cap_{\alpha \in A} E_\alpha^c. \tag{2.23}$$

- Intersection of finitely many open sets. Denote these sets by $E_1$, ..., $E_n$. We show that for every $x \in \cap_{i=1}^n E_i$, there is $(a, b) \subseteq \cap_{i=1}^n E_i$ such that $x \in (a, b)$.

  As $E_1$ is open, there is $(a_1, b_1) \subseteq E_1$ with $x \in (a_1, b_1)$;

  As $E_2$ is open, there is $(a_2, b_2) \subseteq E_2$ with $x \in (a_2, b_2)$;

  Doing this for all $E_i$, we obtain $(a_i, b_i) \subseteq E_i$ such that $x \in (a_i, b_i)$.

  Now set $a = \max\{a_1, ..., a_n\}$ and $b = \min\{b_1, ..., b_n\}$. We claim that $a < x < b$. Since $x \in (a_i, b_i)$, we have $a_i < x < b_i$ for all $i = 1, ..., n$. Therefore $a = \max\{a_1, ..., a_n\} < x < \min\{b_1, ..., b_n\} = b$. Thus we have $(a, b)$ is an open interval and $x \in (a, b)$. Finally, as $a \geqslant a_i, b \leqslant b_i$, we have $(a, b) \subseteq (a_i, b_i) \subseteq E_i$ for all $i = 1, 2, ..., n$. Therefore $(a, b) \subseteq \cap_{i=1}^n E_i$.
- Union of (could be infinitely many) open sets. Denote these sets by $E_\alpha$ with $\alpha \in A$ an index set. Take any $x \in \cup_{\alpha \in A} E_\alpha$. By definition there is $E_{\alpha_0}$ such that $x \in E_{\alpha_0}$. Since $E_{\alpha_0}$ is open, there is $a, b \in \mathbb{R}$ such that $x \in (a, b) \subseteq E_{\alpha_0} \subseteq \cup_{\alpha \in A} E_\alpha$. □

**Remark 2.26.** Note that "union of open sets is open" and "intersection of closed set" can involve infinitely many sets. For example $(1/x, \infty)$ is open for $x > 0$. Then we know $\cup_{x>0}(1/x, \infty)$ is open. On the other hand, the intersection of infinitely many open sets may be closed, for example $\cap_{n \in \mathbb{N}}(-1/n, 1/n)$; The union of infinitely many closed sets may be open, for example $\cup_{n \in \mathbb{N}}\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. Of course it may also be half-open-half-closed.

**Remark 2.27.** From the above theorem we see why it is a good idea to say $\mathbb{R}$ and $\varnothing$ are both open and closed.

**Theorem 2.28. (Structure of open sets)** *Let $E \subseteq \mathbb{R}$ be open. Then there are $a_i, b_i \in \mathbb{R}$, $i \in \mathbb{N}$ such that $E = \cup_{i \in \mathbb{N}}(a_i, b_i)$.*

**Proof.** The proof is beyond the level of this course and is omitted. □

### 2.2.3. Nested sets.

**Definition 2.29. (Nested sets)** *A sequence of sets $\{I_n\}_{n \in \mathbb{N}}$ is said to be nested if*

$$I_1 \supseteq I_2 \supseteq \cdots \tag{2.24}$$

**Theorem 2.30. (Nested interval)** *If $I_n = [a_n, b_n]$ with $a_n, b_n \in \mathbb{R}$ is nested, then $\cap_{n=1}^\infty I_n$ is not empty. If $b_n - a_n \longrightarrow 0$, the intersection is a single point.*

**Proof.** Since $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$, we have $a_n \geqslant a_{n-1}$ and $b_n \leqslant b_{n-1}$. Thus $a_n$ is increasing while bounded above by $b_1$, and $b_n$ is decreasing while bounded below by $a_1$. Therefore there are $a, b \in \mathbb{R}$ such that $a_n \longrightarrow a$, $b_n \longrightarrow b$ as $n \longrightarrow \infty$.

By comparison theorem, $a \leqslant b$. Since $a_n \leqslant a \leqslant b \leqslant b_n$, we have $[a, b] \subseteq [a_n, b_n]$ for every $n$. Therefore $[a, b] \subseteq \cap_{n=1}^{\infty} I_n$.

In particular, if $b_n - a_n \longrightarrow 0$, we have $b = \lim_{n \longrightarrow \infty} b_n = \lim_{n \longrightarrow \infty} a_n + \lim_{n \longrightarrow \infty} (b_n - a_n) = a + 0 = a$. From the above we know that $a \in \cap_{n=1}^{\infty} I_n$. We show that it is the only point in $\cap_{n=1}^{\infty} I_n$. Take any $\tilde{a} \in \cap_{n=1}^{\infty} I_n$. Then we have $a_n \leqslant \tilde{a} \leqslant b_n$. Comparison Theorem then gives $a = \lim_{n \to \infty} a_n \leqslant \lim_{n \to \infty} \tilde{a} \leqslant \lim_{n \to \infty} b_n = a$. Therefore $\tilde{a} = a$ and the proof ends. □

**Remark 2.31.**

- $a_n, b_n \in \mathbb{R}$ is necessary. Otherwise we can take $I_n = [a_n, \infty)$ with $a_n \longrightarrow \infty$ which leads to $\cap_{n=1}^{\infty} I_n \neq \varnothing$.
- It is also necessary that the intervals are closed. Counter-examples are $I_n = (0, 1/n)$, or $I_n = (0, 1/n]$.