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Numbers

> "...my daughter have been studying (Chemistry) at the University for several semesters already and think that they have learned the differential and integral calculus in College; and yet they still don't know why
>
> $$x \cdot y = y \cdot x.\text{"}$$
>
> —– Edmund Landau, Berlin, December 28, 1929.[1]

Main references

- **(Landau:Foundation)** Landau, Edmund, *Foundations of Analysis*, translated by F. Steinhardt, AMS Chelsea Publishing, 3rd edition, 1966. Chapters 1 – 4.

- **(Baby Rudin)** Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill Companies, Inc., 3rd edition, 1976. Chapter 1.

Natural Numbers

Notation. *We will use* $\mathbb{N}$ *to denote natural numbers.* [2]

1. Preface for the student, *Foundations of Analysis*, translated by F. Steinhardt, AMS Chelsea Publishing, 3ed, 1966.

Definition of $\mathbb{N}$

- Axiom 1. $1 \in \mathbb{N}$;
- Axiom 2. For each $x \in \mathbb{N}$ there exists exactly one $x' \in \mathbb{N}$, called the "successor" of x;
- Axiom 3. $\forall x \in \mathbb{N}$, $x' \neq 1$;
- Axiom 4. If $x' = y'$ then $x = y$;
- Axiom 5. (Axiom of Induction): Let $\mathfrak{M} \subseteq \mathbb{N}$ satisfy

  1. $1 \in \mathfrak{M}$;
  2. If $x \in \mathfrak{M}$ then $x' \in \mathfrak{M}$.

  Then $\mathfrak{M} = \mathbb{N}$.

Remark 1. John von Neumann suggested the following construction of $\mathbb{N}$:

Define

$$1 := \{\varnothing\}, \qquad 2 := 1 \cup \{1\}, \qquad 3 := 2 \cup \{2\}, \ldots \tag{1}$$

Note that this does not establish the existence of $\mathbb{N}$. The existence of $\mathbb{N}$ is in fact an axiom:

We accept that there is at least one set S satisfying

i. $1 \in S$;

ii. $x \in S \Longrightarrow x \cup \{x\} \in S$.

Now let W be a collection of all such sets. Define $\mathbb{N} := \cap_{S \in W} S$.

Theorem 2. *$x' \neq x$.*

Proof. Let $\mathfrak{M} := \{x \in \mathbb{N} |\, x' \neq x\}$. Then by Axiom 3, $1 \in \mathfrak{M}$. Now we show that $x \in \mathfrak{M} \Longrightarrow x' \in \mathfrak{M}$. Once this is done the conclusion follows from Axiom 5.

Assume there is $x \in \mathfrak{M}$ such that $x' \notin \mathfrak{M}$. That is $x' \neq x$, but $(x')' = x'$. However by Axiom 4 we have $(x')' = x' \Longrightarrow x' = x$. Contradiction. $\square$

Lemma 3. *If $x \neq y$ then $x' \neq y'$.*

Exercise 1. Prove Lemma 3.

Lemma 4. *If $x \neq 1$, then there is exactly one u such that $u' = x$.*

Exercise 2. Prove Lemma 4 (Hint: Let $\mathfrak{M} = \{1\} \cup \{\text{all numbers with this property}\}$.

2. *There is much debate whether 0 should be included in natural numbers. My personal opinion is that 0 is definitely not as "natural" as 1,2,3,... and therefore shouldn't be included. Thus in this note 0 does not belong to $\mathbb{N}$.*

Addition

We need to define $x+y$ for every pair of $x, y \in \mathbb{N}$.

- First define this for $y=1$: $x+1 := x'$;
- Now assume that this is done for y. We define $x+y' := (x+y)'$. This way addition is defined for each ordered pair (x, y).

Theorem 5. *The way of defining $x+y$ such that $x+1=x'$, $x+y'=(x+y)'$ is unique.*

Proof. Let $+, \oplus$ be two ways of defining addition. Fix an arbitrary $x \in \mathbb{N}$, let $\mathfrak{M} = \{y \in \mathbb{N} | \, x+y = x \oplus y\}$. Then $1 \in \mathfrak{M}$. Now for every $y \in \mathfrak{M}$, we have

$$x+y' = (x+y)' = (x \oplus y)' = x \oplus y'. \tag{2}$$

By Axiom of induction $\mathfrak{M} = \mathbb{N}$. Thus such definition, if it exists, is unique.

Similarly we can prove that such definition indeed exists. Left as exercise. □

Exercise 3. Use induction to prove that $x+y$ can be defined for all $x, y \in \mathbb{N}$.

Theorem 6. $(x+y)+z = x+(y+z)$.

Proof. For any $x, y \in \mathbb{N}$, let $\mathfrak{M} := \{z \in \mathbb{N} | \, (x+y)+z = x+(y+z)\}$. Then we check

$$(x+y)+1 = (x+y)' = x+y' = x+(y+1) \tag{3}$$

so $1 \in \mathfrak{M}$.

For every $z \in \mathfrak{M}$, we have

$$(x+y)+z' = [(x+y)+z]' = [x+(y+z)]' = x+(y+z)' = x+(y+z'). \tag{4}$$

Thus ends the proof. □

Theorem 7. $x+y = y+x$.

Proof. Fix any $y \in \mathbb{N}$. First we prove $1+y = y+1$. Let $\mathfrak{M} := \{y \in \mathbb{N} | \, 1+y = y+1\}$. We have $1+1 = 1+1$ so $1 \in \mathfrak{M}$. Now if $y \in \mathfrak{M}$, we check

$$1+y' = 1+(y+1) = (1+y)+1 = (y+1)+1 = y'+1 \tag{5}$$

so $y' \in \mathfrak{M}$ too. Consequently $1+y = y+1$ for all $y \in \mathbb{N}$.

Now we prove that if $x+y = y+x$, then $x'+y = y+x'$. We have

$$x'+y = (x+1)+y = x+(1+y) = x+(y+1) = (x+y)+1 = (y+x)+1 = y+(x+1) = y+x'. \tag{6}$$

Thus ends the proof. □

Lemma 8. *If $y \neq z$ then $x+y \neq x+z$. Or equivalently $x+y = x+z \Longrightarrow y = z$.*

Exercise 4. Prove Lemma 8.

Ordering

Theorem 9. *For any $x, y \in \mathbb{N}$, exactly one of the following is true:*

i. $x = y$;

ii. *There is exactly one $u \in \mathbb{N}$ such that $x = y + u$;*

iii. *There is exactly one $v \in \mathbb{N}$ such that $y = x + v$.*

Proof.

First we prove that for any $x, y \in \mathbb{N}$, at most one of the three holds. Three cases:

- $x = y$ and $x = y + u$ hold. Then we have $y = y + u \Longrightarrow y + 1 = y' = (y + u)' = y + u'$ which by Lemma 8 gives $1 = u'$ which contradicts Axiom 3.

- $x = y$ and $y = x + v$ hold. Similar to the above case.

- $x = y + u$ and $y = x + v$ hold. Then we have $x + v = (y + u) + v = y + (u + v)$ which by similar argument as above contradicts Axiom 3.

Fix an arbitrary $x \in \mathbb{N}$. We prove that for any $y \in \mathbb{N}$, at least one of the above holds.

Let $\mathfrak{M} := \{y \in \mathbb{N} |$ exactly one of $x = y, x = y + u, y = x + v\}$ holds.

- $1 \in \mathfrak{M}$. There are two cases.

  ◦ $x = 1$. Then $x = y$.

  ◦ $x \neq 1$. Then by Lemma 4 there is $u \in \mathbb{N}$ such that $x = u'$. Thus $x = u + 1 = 1 + u = y + u$.

- If $y \in \mathfrak{M}$ then $y' \in \mathfrak{M}$. There are three cases.

  ◦ $x = y$. Then $y' = y + 1 = x + 1$ and therefore $y' \in \mathfrak{M}$.

  ◦ $x = y + u$. Then there are two cases.

    – $u = 1$. Then $x = y'$.

    – $u \neq 1$. Then by Lemma 4 there is $v \in \mathbb{N}$ such that $u = v'$. This gives

    $$x = y + u = y + v' = (y + v)' = y' + v. \tag{7}$$

    So $y' \in \mathfrak{M}$.

  ◦ $y = x + v$. Then

  $$y' = (x + v)' = x + v' \tag{8}$$

  and $y' \in \mathfrak{M}$.

Thus $y' \in \mathfrak{M}$ and the proof ends. ∎

Definition 10. (Ordering) *If $x = y + u$, denote $x > y$; If $y = x + v$, denote $x < y$.*

Theorem 11. *For any given x, y, we have exactly one of the following: $x = y, x < y, x > y$.*

Exercise 5. If $x < y$ then $y > x$.

Definition 12. *Define $x \geqslant y$ as $x > y$ or $x = y$. Define $x \leqslant y$ as $x < y$ or $x = y$.*

Exercise 6. If $x \leqslant y$ then $y \geqslant x$; If $x \geqslant y$ then $y \leqslant x$.

Exercise 7. If $x \leqslant y$ and $y \leqslant x$ then $x = y$.

Exercise 8. If $x < y, y < z$ then $x < z$; If $x \leqslant y, y < z$ then $x < z$; If $x < y, y \leqslant z$ then $x < z$; If $x \leqslant y, y \leqslant z$ then $x \leqslant z$.

Exercise 9. If $x > y, z > u$ then $x + z > y + u$.

Exercise 10. If $x < y$, then $x + 1 \leqslant y$.

Theorem 13. *Let $A \subseteq \mathbb{N}$ be nonempty. Then there is a unique least element, that is $a \in A$ such that for all $b \in A$, $a \leqslant b$.*

Proof.

- If $1 \in A$ then 1 is the least element, since for all $x \in \mathbb{N}$, $x \neq 1$, there is u such that $x = u' = 1 + u > 1$.

- If $1 \notin A$, let $\mathfrak{M} := \{x \in \mathbb{N} |\ \forall b \in A, x \leqslant b\}$. Now if for every $x \in \mathfrak{M}$ we have $x + 1 \in \mathfrak{M}$, by the Axiom of induction $\mathfrak{M} = \mathbb{N}$ and $A = \varnothing$. Contradiction. Thus there is $a \in \mathfrak{M}$ satisfying:

$$a \in \mathfrak{M}, a + 1 \notin \mathfrak{M}. \tag{9}$$

  We claim $a \in A$. Since otherwise, by definition of $\leqslant$, for every $b \in A$ there must hold $a < b$ which implies $a + 1 \leqslant b$. Consequently $a + 1 \in \mathfrak{M}$. Contradiction.

  Now $a \in \mathfrak{M} \Longrightarrow \forall b \in A, a \leqslant b$ so a is a least element. Uniqueness follows from $a \leqslant b, b \leqslant a \Longrightarrow a = b$. □

Multiplication

Theorem 14. *To every pair of $x, y \in \mathbb{N}$, we can assign in exactly one way a $z \in \mathbb{N}$, denoted $x \cdot y$ (or $x\,y$ when no confusion may arise), such that*

i. $x \cdot 1 = x$ *for every* x;

ii. $x \cdot y' = x \cdot y + x$ *for every* x *and every* y.

Proof. Left as exercise. □

Exercise 11. $x \cdot y = y \cdot x$.

Exercise 12. $x\,(y + z) = x\,y + x\,z$.

Exercise 13. $(x\,y)\,z = x\,(y\,z)$.

Exercise 14. If $x > y (= y, < y)$ then $x\,z > y\,z (= y\,z, < y\,z)$. If $x\,y > y\,z (= y\,z, < y\,z)$ then $x > y (= y, < y)$.

Exercise 15. If $x > y, z > u$ then $x\,z > y\,u$.

(Positive) Rational Numbers

Definition

Definition 15. *Consider all ordered pairs* (x_1, x_2) *with* $x_1, x_2 \in \mathbb{N}$. *Define the equivalance*

$$(x_1, x_2) \sim (y_1, y_2) \Longleftrightarrow x_1\,y_2 = x_2\,y_1. \tag{10}$$

Theorem 16. $\sim$ *is indeed an equivalence relation, that is*

1. $(x_1, x_2) \sim (x_1, x_2)$;
2. $(x_1, x_2) \sim (y_1, y_2) \Longleftrightarrow (y_1, y_2) \sim (x_1, x_2)$;
3. $(x_1, x_2) \sim (y_1, y_2)$ *and* $(y_1, y_2) \sim (z_1, z_2)$ *then* $(x_1, x_2) \sim (z_1, z_2)$.

Proof. Exercise. $\square$

Exercise 16. $(x_1, x_2) \sim (x_1\,x, x_2\,x)$ for any $x \in \mathbb{N}$.

Notation. *From now on we will denote* (x_1, x_2) *by* $\frac{x_1}{x_2}$, *or* x_1/x_2.

Definition 17. $\frac{x_1}{x_2} > \frac{y_1}{y_2}$ *if and only if* $x_1\,y_2 > y_1\,x_2$. $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *if and only if* $\frac{y_1}{y_2} > \frac{x_1}{x_2}$.

Theorem 18. *Exactly one of the following three holds:*

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}, \qquad \frac{x_1}{x_2} > \frac{y_1}{y_2}, \qquad \frac{x_1}{x_2} < \frac{y_1}{y_2}. \tag{11}$$

Exercise 17. If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} < \frac{z_1}{z_2}$ then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Definition 19. $\frac{x_1}{x_2} \geqslant \frac{y_1}{y_2}$ *if and only if* $\frac{x_1}{x_2} > \frac{y_1}{y_2}$ *or* $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$. $\frac{x_1}{x_2} \leqslant \frac{y_1}{y_2}$ *if and only if* $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *or* $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.

Exercise 18. If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \leqslant \frac{z_1}{z_2}$ then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Exercise 19. If $\frac{x_1}{x_2} \geqslant \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \geqslant \frac{x_1}{x_2}$, then $\frac{x_1}{x_2} = \frac{y_1}{y_2}$.

Theorem 20. *If* $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *then there is* $\frac{z_1}{z_2}$ *such that* $\frac{x_1}{x_2} < \frac{z_1}{z_2} < \frac{y_1}{y_2}$.

Proof. Set

$$z_1 = x_1 + y_1, \qquad z_2 = x_2 + y_2. \tag{12}$$

Then it satisfies the requirement. $\square$

Problem 1. Define addition, subtraction, multiplication, division, and justify their properties.

Definition 21. *We define a positive rational number X to be an equivalence class of all pairs $\frac{y_1}{y_2}$ equivalent to a fixed pair $\frac{x_1}{x_2}$. We denote the set of positive rational numbers $\mathbb{Q}^+$.*

Definition 22. *Two rational numbers X, Y are equal, denoted $X = Y$, if there are $\frac{x_1}{x_2} \in X$ and $\frac{y_1}{y_2} \in Y$ such that $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.*

Remark 23. We can easily define $X < Y, X \leqslant Y, X > Y, X \geqslant Y$.

Theorem 24. *Let X, Y be rational numbers. Then there is an natural number z such that $z X > Y$.*

Proof. Take $\frac{x_1}{x_2} \in X, \frac{y_1}{y_2} \in Y$. We need to prove that there is $z \in \mathbb{N}$ such that $z x_1 y_2 > x_2 y_1$. More generally, we prove that for any $x, y \in \mathbb{N}$, there is z such that $z x > y$.

Fix x, Let $\mathfrak{M} := \{y \in \mathbb{N} | \exists z \in \mathbb{N}, z x > y\}$.

- $1 \in \mathfrak{M}$. Two cases.
  - $x = 1$. Then we have $1' \cdot x = x + x > x = 1$.
  - $x \neq 1$. Then there is $u \in \mathbb{N}$ such that $x = u' = 1 + u > 1$.

  Therefore $1 \in \mathfrak{M}$.

- $y \in \mathfrak{M}$ then $y' \in \mathfrak{M}$.

  Since $y \in \mathfrak{M}$ there is $z \in \mathbb{N}$ such that $z x > y$. Thus there is $u \in \mathbb{N}$ such that $z x = y + u$. Now consider $z' x = z x + x = y + u + x$. It is easy to see that $u + x \neq 1$ thus there is $v \in \mathbb{N}$ such that $u + x = v'$. This gives

$$z' x = y + u + x = y + v' = y' + v > y'. \tag{13}$$

Thus ends the proof. □

Exercise 20. Let $x, y \in \mathbb{N}$. Prove that there is $v \in \mathbb{N}$ such that $x + y = v'$.

Theorem 25. *Define $S := \left\{X \in \mathbb{Q}^+ | \ X \sim \frac{x}{1} \text{ for some } x \in \mathbb{N}\right\}$. Denote $\frac{1}{1}$ by 1. For any $X \in S$, define $X' := X + \frac{1}{1}$. Then S satisfies Axioms 1 – 5 of $\mathbb{N}$ and is therefore $\mathbb{N}$.*

Notation. *In the following we will stop using capital letters to denote rational numbers.*

Dedekind cuts

Definition 26. *A subset $\xi \subseteq \mathbb{Q}$ is called a "cut" if and only if*

i. it contains a rational number, but does not contain all rational numbers;

ii. every rational number in the set is smaller than every rational number not in the set;

iii. it does not contain a greatest rational number.

Exercise 21. Let $\xi \subseteq \mathbb{Q}$ be a cut. Prove that

$$[x \in \xi] \Longrightarrow [\forall y < x, y \in \xi]; \qquad [x \notin \xi] \Longrightarrow [\forall y > x, y \notin \xi]. \tag{14}$$

Exercise 22. Let $x \in \mathbb{Q}$. Then $\xi := \{y \in \mathbb{Q} | \, y < x\}$ is a cut.

Definition 27. *Two cuts ξ, η are said to be equal, denoted $\xi = \eta$, if they are equal as sets.*

Theorem 28. *"$=$" is an equivalence relation, that is*

$$\xi = \xi; \qquad \xi = \eta \Longrightarrow \eta = \xi; \qquad \xi = \eta, \eta = \zeta \Longrightarrow \xi = \zeta. \tag{15}$$

Definition 29. *Let ξ, η be two cuts. Say $\xi < \eta$ if and only if $\xi \subsetneq \eta$. Say $\xi > \eta$ if and only if $\eta < \xi$. Say $\xi \leqslant \eta$ if $\xi \subseteq \eta$ and $\xi \geqslant \eta$ if $\eta \leqslant \xi$.*

Exercise 23. For any ξ, η exactly one of the following is true: $\xi = \eta; \xi > \eta; \xi < \eta$.

Exercise 24. If $\xi \leqslant \eta$ and $\eta \leqslant \xi$, then $\xi = \eta$.

Addition of cuts

Theorem 30. *Let ξ, η be cuts. Then*

$$\zeta := \{x + y | \, x \in \xi, y \in \eta\} \tag{16}$$

is also a cut.

Proof. Since ξ, η are cuts they are not empty. Thus ζ is not empty. On the other hand, ξ, η are not $\mathbb{Q}$, therefore there is $a, b \in \mathbb{Q}$ such that $a \notin \xi, b \notin \eta$. By definition of cuts we have

$$\forall x \in \xi, x < a; \qquad \forall y \in \eta, y < b. \tag{17}$$

This gives

$$\forall z \in \zeta, z < a + b. \tag{18}$$

Therefore ζ is not $\mathbb{Q}$.

Next we prove that if $z \in \zeta$ then all $z' < z$ also belongs to ζ. Let $z \in \zeta$. Then there are $x \in \xi, y \in \eta$ such that $z = x + y$. Now for any $z' < z$, we have $\frac{z'}{z} x < x \Longrightarrow \frac{z'}{z} x \in \xi$ and similarly $\frac{z'}{z} y \in \eta$. Now we have

$$\frac{z'}{z} x + \frac{z'}{z} y = z'. \tag{19}$$

Now we prove that if $z \notin \zeta$ and $z' > z$, then $z' \notin \zeta$. This is obvious through proof by contradiction and what we have just proved.

Finally since neither ξ nor η has a greatest number, for any $z \in \zeta$ there is always $z' \in \zeta$ satisfying $z' > z$. $\square$

Definition 31. *Let ξ, η be cuts. Then define their sum to be*

$$\xi + \eta := \{x + y | \, x \in \xi, y \in \eta\}. \tag{20}$$

Exercise 25. Prove $\xi + \eta > \xi$.

Exercise 26. Prove $\xi + \eta = \eta + \xi$.

Exercise 27. Prove $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$.

Exercise 28. Let ξ be a cut. Let $a \in \mathbb{Q}^+$. Then there are $x \in \xi, y \notin \xi$ such that $y - x = a$. (Hint: Consider $x + n\,a$ for $n \in \mathbb{N}$)

Theorem 32. *If $\xi > \eta$, then there is a unique ζ such that $\eta + \zeta = \xi$.*

Proof. Uniqueness is trivial. We prove existence. Since $\xi > \eta$, the set

$$A := \{x \in \mathbb{Q} |\, x \in \xi, x \notin \eta\} \tag{21}$$

is not empty. Define

$$\zeta := \{x - y |\, x, y \in A, x > y\}. \tag{22}$$

We first prove that it is a cut. (Need to identify $x \in \mathbb{Q}$ with a cut?)

i. Since A is non... □

Definition 33. *Denote that ζ in the above theorem as $\xi - \eta$.*

Multiplication of cuts

Theorem 34. *Let ξ, η be cuts. Then*

$$\zeta := \{x\,y |\, x \in \xi, y \in \eta\} \tag{23}$$

is also a cut.

Proof. Left as exercise. □

Definition 35. *The ζ above is called the product of ξ, η, denoted $\xi \cdot \eta$ (or $\xi\,\eta$ if no confusion arises)*

Exercise 29. $\xi\,\eta = \eta\,\xi$;

Exercise 30. $(\xi\,\eta)\,\zeta = \xi\,(\eta\,\zeta)$;

Exercise 31. $\xi\,(\eta + \zeta) = \xi\,\eta + \xi\,\zeta$.

Exercise 32. If $\xi > \eta$, then $\xi\,\zeta > \eta\,\zeta$.

Theorem 36. *Let $\mathbf{1}$ be the cut $\{x \in \mathbb{Q} |\, x < 1\}$. Then for any cut ξ, we have $\mathbf{1} \cdot \xi = \xi \cdot \mathbf{1} = \xi$.*

Proof. Exercise. □

Theorem 37. *For any ξ, there is a unique η such that $\xi\,\eta = \mathbf{1}$.*

Proof. Take

$$\eta := \left\{\frac{y}{z} |\, y \in \mathbf{1}, z \notin \xi\right\}. \tag{24}$$

Then η is the cut we need. $\square$

Exercise 33. Finish the above proof.

Exercise 34. ξ, η are cuts. Then there is a unique v such that $\xi\, v = \eta$.

Exercise 35. Define division.

Problem 2. Let $x \in \mathbb{Q}$. Define $\xi_x := \{y \in \mathbb{Q}|\ y < x\}$. Then it is a cut. Prove that such "rational cuts" satisfy the same arithmetic rules as rational numbers.

Problem 3. Let $n \in \mathbb{N}$. Defind $\xi_n := \{y \in \mathbb{Q}|\ y < n\}$. Then it is a cut. Prove that such "integral cuts" satisfy the axioms for natural numbers.

Problem 4. If $\xi < \eta$, then there is a rational number Z such that $\xi < Z < \eta$.

Exercise 36. For each ζ, the equation $\xi^2 := \xi \cdot \xi = \zeta$ has exactly one solution.

Definition 38. *Any cut which is not a rational number is called an irrational number.*

Theorem 39. *There exists an irrational number.*

Proof. It suffices to show that the solution to $\xi^2 = 1'$ (recall that this is the successor of 1) is irrational.

Otherwise we would have $\xi = \frac{x}{y}$ where $x,\ y \in \mathbb{N}$. Take $x,\ y$ such that y is smallest. We easily see that $y < x < 1'\, y$. Set $x - y = u$, then $u < y$. Next set $y - u = t$. Then

$$\begin{aligned} x\,x + t\,t &= (y + u)\,(y + u) + t\,t \\ &= (y\,y + 1'\,y\,u) + (u\,u + t\,t) \\ &= (y\,y + 1'\,u\,(u + t)) + (u\,u + t\,t) \\ &= (y\,y + 1'\,(u\,u)) + (u\,u + 1'\,u\,t + t\,t) \\ &= y\,y + 1'\,u\,u + y\,y \\ &= 1'\,y\,y + 1'\,u\,u \\ &= x\,x + 1'\,u\,u. \end{aligned} \tag{25}$$

Thus

$$\frac{t}{u} \cdot \frac{t}{u} = 1'. \tag{26}$$

But $u < y$. Contradiction. $\square$

Real numbers

Definition of real numbers

Definition 40. *We call any cut ξ a "positive real number".*

Notation. *As in the following we will not use the idea of "cut" anymore, from now on we will simply denote real numbers by any convenient letters (English or Greek).*

Definition 41. *Let the symbol 0 be defined through the property: For any positive real number x, $0 + x = x + 0 = x$.*

Exercise 37. Prove that 0 is different from any positive real number defined in the previous section.

Definition 42. *For any positive real number x, let the symbol $-x$ be defined through the property*

$$(-x)+x=x+(-x)=0. \tag{27}$$

These numbers are called "negative numbers".

Exercise 38. Prove that $-(x+y)=(-x)+(-y)$.

Definition 43. *Positive real numbers, negative real numbers, and 0 together are called "real numbers".*

Definition 44. *We extend the operation "$-$" as follows.*

- *If $x>0$, $-x$ is already defined;*
- *If $x=0$, $-x:=0$;*
- *If $x<0$, then there is ξ such that $x=-\xi$, define $-x:=\xi$.*

Order

Definition 45. *Let x,y be real numbers. Then we say $x>y$ if and only if*

- *x,y positive $x>y$, or*
- *x,y negative, $-x<-y$, or*
- *$x=0$, y negative, or*
- *$y=0$, x positive, or*
- *x positive, y negative.*

We say $x<y$ if and only if $y>x$.

Exercise 39. For any $x,y\in\mathbb{R}$, exactly one of the following holds:

$$x=y, \quad x>y, \quad x<y. \tag{28}$$

Exercise 40. $x>0$ if and only if it is a positive real number.

Definition 46. *Let x,y be real numbers. Then we say $x=y$ if and only if*

- *x,y positive, $x=y$, or*
- *$x=y=0$, or*
- *x,y negative, $-x=-y$.*

Exercise 41. Prove that $=$ is an equivalence relation.

Definition 47. *Let x, y be real numbers. We say $x \geqslant y$ if and only if $x > y$ or $x = y$. We say $x \leqslant y$ if and only if $y \geqslant x$.*

Addition and subtraction

Definition 48. *Let x, y be real numbers. We define their sum $x + y$ as follows.*

- *If $x, y > 0$, already defined;*
- *If $x = 0$, $x + y = y$; If $y = 0$, $x + y = x$.*
- *If $x > 0, y < 0$, consider three sub-cases:*
  - *If $x > (-y)$, then define $x - y := u$ where u is the unique solution to $(-y) + u = x$.*
  - *If $x = \eta$, then by definition of negative numbers $x - y := 0$.*
  - *If $x < (-y)$, then define $x - y := -u$ where u is the unique solution to $x + u = (-y)$.*
- *If $x < 0$, $y > 0$ the definition is similar.*

Exercise 42. Prove that $x + y = y + x$.

Exercise 43. Define the absolute value function as

$$|x| := \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}. \tag{29}$$

Prove that $|-x| = |x|$.

Exercise 44. Prove $|x + y| \leqslant |x| + |y|$.

Definition 49. *Let x, y be real numbers. We define their difference $x - y := x + (-y)$.*

Exercise 45. Prove that $x - y = -(y - x)$.

Exercise 46. Prove that $x - y > 0 \Longleftrightarrow x > y$.

Exercise 47. Prove that for every $x \in \mathbb{R}$, there are $y, z \in \mathbb{R}^+$ such that $x = y - z$.

Exercise 48. Prove that $(x + y) + z = x + (y + z)$.

Multiplication and division

Definition 50. *Let $x, y \in \mathbb{R}$. We define their product $x \cdot y$ (or $x\, y$ when no confusion arises) through*

- *If $x, y > 0$, already defined;*
- *If $x = 0$ or $y = 0$, $x \cdot y := 0$;*
- *If $x, y < 0$, $x \cdot y := (-x) \cdot (-y)$;*
- *If $x > 0, y < 0$, $x \cdot y := -(x \cdot (-y))$;*
- *If $x < 0, y > 0$, $x \cdot y := -((-x) \cdot y)$.*

Exercise 49. If $x\,y=0$, then at least one of x, y is 0.

Exercise 50. $|x\,y|=|x|\cdot|y|$.

Exercise 51. $x\,y=y\,x$.

Exercise 52. $x\cdot 1=x$.

Exercise 53. $(-x)\,(-y)=x\,y$.

Exercise 54. $(x\,y)\,z=x\,(y\,z)$.

Exercise 55. $x\,(y+z)=x\,y+x\,z$.

Exercise 56. There is a unique u solving $x\,u=y$ if $x\neq 0$.

Theorem 51. $x, y\in\mathbb{R}$. *Then* $x\,y=(-x)\,(-y)$.

Proof. We have

$$(x+(-x))\,y=0\Longrightarrow x\,y+(-x)\,y=0. \tag{30}$$

On the other hand

$$(-x)\,(y+(-y))=0\Longrightarrow(-x)\,y+(-x)\,(-y)=0. \tag{31}$$

Therefore $x\,y=(-x)\,(-y)$. $\square$

Recall that for any $x\in\mathbb{R}^+$, there is a unique y such that $x\,y=1$. From the above theorem we see that $(-y)$ is the unique solution to $(-x)\,(-y)=1$. Thus division can be easily defined for all real numbers (denominator $\neq 0$).

Dedekind's fundamental theorem

Theorem 52. *Let* $A\cup B=\mathbb{R}$ *with*

i. A, B *not empty;*

ii. $A\cap B=\varnothing$*;*

iii. $\forall x\in A, y\in B,\ x<y$.

Then there is exactly one real number z *such that* $\{x|\,x<z\}\subseteq A, \{x|\,x>z\}\subseteq B$.

Proof.

- Uniqueness.

  If there are $z_1\neq z_2$, then wlog $z_1<z_2$. Then $\frac{z_1+z_2}{2}\in A\cap B$. Contradiction.

- Existence.

  1. $A\cap\mathbb{R}^+\neq\varnothing$. Consider $A\cap\mathbb{Q}$. If there is $x=\max\,(A\cap\mathbb{Q})$, set $\xi=A\cap\mathbb{Q}^+-\{x\}$. Otherwise set $\xi=A\cap\mathbb{Q}^+$.

     Now it can be check that ξ is a cut and therefore determine a real number z.

For any $x < z$, if $x < 0$, since $0 < z$ we have $x \in A$; If $x > 0$ we have $x < \frac{x+z}{2} < z$ and is therefore in A.

Similarly we can show $\{x \mid x > z\} \subseteq B$.

2. $A \cap \mathbb{R}^+ = \varnothing$, $0 \in A$. Then we prove that $z = 0$.

3. Other cases are similarly discussed. □

Properties of real numbers

$\mathbb{R}$ is an ordered field

Definition 53. (Field) *A set F is called a field if there are two functions defined: $\oplus, \odot: F \times F \mapsto F$, satisfying the following:*

- *Axioms for addition:*

  i. $x \in F, y \in F \Longrightarrow x \oplus y \in F$*;*

  ii. $x \oplus y = y \oplus x$*;*

  iii. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$*;*

  iv. There is an element 0 *satisfying* $0 \oplus x = x$ *for any* $a \in F$*;*

  v. For each $x \in F$*, there is an element* $y \in F$ *such that* $y \oplus x = 0$*. Denote it by* $-x$*.*

- *Axioms for multiplication:*

  a) $x \in F, y \in F \Longrightarrow x \odot y \in F$*;*

  b) $x \odot y = y \odot x$*;*

  c) $x \odot (y \odot z) = (x \odot y) \odot z$*;*

  d) There is an element $i \in F$ *such that* $i \odot x = x$ *for every* $x \in F$*. Denote it by* 1*;*

  e) For every $x \in F$*, there is a* $y \in F$ *such that* $x \odot y = 1$*. Denote* y *by* x^{-1}*.*

- *The distributive law:*

  A) For every $x, y, z \in F$*,* $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$*.*

Exercise 57. Prove the uniqueness of the special elements $0, 1$. Also prove that for each $x \in F$ and each $y \in F, y \neq 0$, $-x$ and y^{-1} are unique.

Exercise 58. Give a reasonable definition to the new function $F \times F \times F \mapsto F$ which we hope to denote by $x \oplus y \oplus z$ and justify your definition.

Exercise 59. Let A be a set. Let $W := \{\text{subsets of } A\}$. Define addition and multiplication on W as

$$x \oplus y := x \cup y; \qquad x \odot y := x \cap y. \tag{32}$$

Does this make W a field? Justify your answer.

Remark 54. Clearly we can further define "subtration" and "division" through

$$x - y := x + (-y); \qquad x/y := x\,(y^{-1}). \tag{33}$$

Example 55. Prove that $4 = 2 \oplus 2$.

Proof. By definition,

$$4 = 3 \oplus 1 = (2 \oplus 1) \oplus 1 = 2 \oplus (1 \oplus 1) = 2 \oplus 2. \tag{34}$$

Thus ends the proof. ☐

Exercise 60. Let $x \in F$. Prove that $x \oplus x \oplus x = 3 \odot x$.

Exercise 61. Prove

a) $x \oplus y = x \oplus z \Longrightarrow y = z$;

b) $x \odot y = x \odot z \Longrightarrow y = z$ unless $x = 0$.

Exercise 62. Denote $x \odot x$ by x^2. Prove that $(x \oplus y)^2 = x^2 \oplus (2 \odot x \odot y) \oplus y^2$.

Exercise 63. Prove the following

a) $x \odot 0 = 0$;

b) $x \odot (-y) = -(x \odot y)$;

c) $(-x) \odot (-y) = x \odot y$;

d) If $x \neq 0$, $(-x)^{-1} = -(x^{-1})$;

Notation. *From now on we will discard $\odot$ and $\oplus$, and simply use the usual notations $x \cdot y$ ($x\,y$), x/y, $x \pm y$.*

Definition 56. (Order) *Let S be a set. An "order" on S is a relation, denoted by $<$, with the following two properties:*

i. If $x \in S, y \in S$ then exactly one of the following is true.

$$x < y, x = y, x > y; \tag{35}$$

ii. If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark 57. $\geqslant, \leqslant$ can be defined in the natural way.

Definition 58. (Ordered field) *F is an ordered field if*

i. It is a field;

ii. It has an order;

iii. The field operations are consistent with the order structure:

- $x, y, z \in F, y < z \Longrightarrow x + y < x + z$;
- $x, y \in F, x > 0, y > 0 \Longrightarrow x\,y > 0$.

Exercise 64. Let F be an ordered field. Let $x, y \in F$. Prove that if $x\,y < 0$, then one is positive and the other negative.

Exercise 65. Let F be an ordered field. Let $x, y, z \in F$. Then

a) If $x > 0$ then $-x < 0$ and vice verse;

b) If $x \neq 0$, then $x^2 > 0$; In particular $1 > 0$;

c) If $x > 0$, $y < z$, then $x\,y < x\,z$;

d) If $0 > x > y$, then $0 > \frac{1}{y} > \frac{1}{x}$.

Theorem 59. *$\mathbb{R}$ as constructed in the previous sections is an ordered field.*

$\mathbb{R}$ has least upper bound property

Definition 60. (Upper bound) *Suppose S is an ordered set, and $E \subseteq S$. If there is a $\beta \in S$ such that $x \leqslant \beta$ for every $x \in E$, we say E is bounded above, and called β an upper bound of E.*

Remark 61. Lower bound can be defined similarly.

Definition 62. (Least upper bound) *Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose that there exists an $\alpha \in S$ such that*

i. α is an upper bound of E;

ii. If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the least upper bound (also called "supreme") of E. Denoted as

$$\alpha = \sup E. \tag{36}$$

Remark 63. Greatest lower bound (or infimum) $\alpha = \inf E$ can be defined similarly.

Definition 64. (LUB property) *An ordered set S is said to have the least-upper-bound (LUB) property if:*

$$E \subseteq S, E \neq \varnothing, E \text{ is bounded above}, \Longrightarrow \sup E \text{ exists in } S. \tag{37}$$

Exercise 66. Prove that $\mathbb{Q}$ does not have LUB property.

Exercise 67. S has least-upper-bound property $\Longleftrightarrow$ S has greatest lower bound property.

Theorem 65. *Let F be an ordered field with LUB property. Then for every $x > 0$, there is a unique $y > 0$ such that $y^3 = x$.*

Proof. That y is unique is trivial since $y_1 < y_2 \Longrightarrow y_1^3 < y_2^3$.

To show existence, consider $A := \{t \in F |\, t \geqslant 0, t^3 < x\}$ and set $y := \sup A$. Note that since $0^3 = 0 < x$, we have $0 \in A$ and therefore y exists. Furthermore taking $z = \min\left\{1, \frac{x}{2}\right\}$ we have

$$z^3 \leqslant z \leqslant \frac{x}{2} < x \Longrightarrow z \in A \Longrightarrow y > 0. \tag{38}$$

Now we show that it cannot hold that $y^3 < x$ or $y^3 > x$. Assume the contrary. Two cases.

- $y^3 < x$. Then $1 < y^{-3}\, x$. If we can find $\varepsilon > 0$ such that $(1 + \varepsilon)^3 < y^{-3}\, x$ then $[(1 + \varepsilon)\, y]^3 < x$ and we have a contradiction.

  Thus it suffices to show that $1 < x \Longrightarrow \exists \varepsilon > 0, (1 + \varepsilon)^3 < x$. We have

  $$(1+\varepsilon)^3 = 1 + 3\,\varepsilon + 3\,\varepsilon^2 + \varepsilon^3 = 1 + (3 + 3\,\varepsilon + \varepsilon^2)\,\varepsilon. \tag{39}$$

  Now take $\varepsilon = \min\left\{1, \frac{x-1}{8}\right\}$. We have

  $$(1+\varepsilon)^3 = 1 + (3 + 3\,\varepsilon + \varepsilon^2)\,\varepsilon \leqslant 1 + 7\,\varepsilon \leqslant 1 + \frac{7}{8}\,(x-1) < x. \tag{40}$$

  Thus we are done.

- $y^3 > x$. In this case all we need is $(1 - \varepsilon)^3 > y^{-3}\, x$. In light of

  $$(1-\varepsilon)^3 = 1 - 3\,\varepsilon + 3\,\varepsilon^2 - \varepsilon^3 > 1 - (3 + \varepsilon^2)\,\varepsilon \tag{41}$$

  the proof is similar to that for the previous case.

Thus $y^3 = x$ and the proof ends. □

Exercise 68. Prove that $y_1 < y_2 \Longrightarrow y_1^3 < y_2^3$ with no assumption on the signs of y_1, y_2.

Exercise 69. Fill in the details for the $y^3 > x$ case.

Exercise 70. Let F be an ordered field with LUB property. Let $\alpha \in \mathbb{Q}$. Define α-th power in the natural way. Then for every $x > 0$, there is a unique $y > 0$ such that $y^\alpha = x$.

Problem 5. Let F be an ordered field with LUB property. Let $\alpha \in \mathbb{R}$. Define x^α for all $x \in F, x > 0$. (Hint: See Problem 6 of Chapter 1 in **(Baby Rudin)**)

Problem 6. Let F be an ordered field with LUB property. Fix $b > 1, y > 0$. Prove that there is a unique $x \in \mathbb{R}$ such that $b^x = y$. (Hint: See Problem 7 of Chapter 1 in **(Baby Rudin)**)

Theorem 66. $\mathbb{R}$ *as constructed in the previous sections has the LUB property.*

Proof. We consider the following cases:

- All the upper bounds are positive. Then $E \cup \mathbb{R}^+$ is not empty. We identify real numbers with cuts and define

  $$\alpha := \cup_{\xi \in E \cup \mathbb{R}^+} \xi. \tag{42}$$

- 0 is an upper bound but there is no negative upper bounds. In this case by definition $\sup E = 0 \in \mathbb{R}$.

- There is at least one negative upper bound. Define

  $$F := \{-\alpha |\, \alpha \text{ is an upper bound for } E,\ \alpha < 0\}. \tag{43}$$

  Now treat member of F as cuts and define $\eta := -\xi$ where

  $$\xi := \cup_{\alpha \in F} \alpha. \tag{44}$$

Clearly $\eta = \sup E$. $\square$

Problem 7. Let F be an ordered field satisfying LUB. Prove that there is $x \in F$ such that $x^2 = 2$.

Exercise 71. Let F be an ordered field satisfying LUB. Let $a \in F$ be $a > 0$. Prove that there is $x \in F$ such that $x^2 = a$.

Archimedean

Definition 67. *A ordered field F is said to be Archimedean if and only if $\mathbb{N}$ does not have an upper bound in F. Here $\mathbb{N}$ is defined as $\{1, 1+1, 1+1+1, \ldots\}$.*

Remark 68. It is obvious that $\mathbb{R}$ is Archimedean.

Theorem 69. *A ordered field F satisfying LUB then Archimedian.*

Proof. Assume $\mathbb{N}$ is bounded from above. Then there is least upper bound $a = \sup \mathbb{N}$. By definition of sup , $a - 1$ is not a upper bound for $\mathbb{N}$. Thus there is $n \in \mathbb{N}$ such that $n > a - 1$. But then $a < n + 1 \in \mathbb{N}$. Contradiction. $\square$

Exercise 72. Find an ordered field that is Archimedean but does not satisfy LUB.

Theorem 70. *An ordered field F is Archimedean $\Longleftrightarrow$ $\mathbb{Q}$ is dense in F.*

Proof.

- $\Longrightarrow$. Take any $x, y \in F, x < y$. We prove that there is $z \in \mathbb{Q}$ such that $x < z < y$. It is clear that it suffices to discuss the situation $0 < x < y$. In this case we need to find $m, n \in \mathbb{N}$ such that

  $$x < \frac{m}{n} < y \Longleftrightarrow n\,x < m < n\,y. \qquad (45)$$

  Since F is Archimedean, there is $n \in \mathbb{N}$ such that $n\,(y - x) > 1$. Fix this n. Consider the set $A{:} = \{k \in \mathbb{N}|\ k < n\,y\}$. Again since F is Archimedean, this set is finite and we take $m = \max A$. We claim that $m > n\,x$. Assume otherwise, then $m < n\,x \Longrightarrow m + 1 < n\,x + n\,(y - x) = n\,y$. We see that $m + 1 \in A$. Contradiction.

- $\Longleftarrow$. Take any positive $y \in F$. We show that it cannot be an upper bound for $\mathbb{N}$.

  Since $\mathbb{Q}$ is dense in F, there is $\frac{m}{n}$ such that

  $$0 < \frac{m}{n} < \frac{1}{y} \Longrightarrow m\,y < n \Longrightarrow y < n. \qquad (46)$$

  Thus F is Archimedean. $\square$

$\mathbb{R}$ is unique

Theorem 71. $\mathbb{R}$ *is the unique ordered Archimedean field. Or equivalently* $\mathbb{R}$ *is the unique field where* $\mathbb{Q}$ *is dense.*