Contents

Note.

- The exercises and problems in this article does not cover every possible topic in the midterm exam.
- You should review homework and lecture notes.
- Please try to work on the exercises and problems before looking at the solutions.

A. Geometry of $\mathbb{R}^N$

1. Exercises

Exercise 1. Consider all real $M \times N$ matrices. Define addition, scalar multiplication as follows:

$$A + B := (a_{ij} + b_{ij}); \tag{1}$$

$$a\,A := (a\,a_{ij}). \tag{2}$$

- Prove that this set becomes linear vector space.
- Define the operation

  $$(A, B) := \mathrm{tr}(A^T B) \tag{3}$$

  where the "trace" is define for all $N \times N$ matrices as

  $$\mathrm{tr}\,A = \sum_{i=1}^{N} a_{ii}. \tag{4}$$

  Is this an inner product? Justify your answer.
- If the above is an inner product, what is the norm defined by it?

Exercise 2. Let $A = (a_{ij}) \in \mathbb{R}^{M \times N}$ and let $\boldsymbol{v} \in \mathbb{R}^N$. Prove

$$\|A\,\boldsymbol{x}\| \leqslant \|A\|_F\, \|\boldsymbol{x}\|. \tag{5}$$

Here $\|\cdot\|$ is the Euclidean norm for vectors defined in class, and $\|\cdot\|_F$ is a matrix norm called Frobenius norm, defined by

$$\|A\|_F := \left(\sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}^2 \right)^{1/2}. \tag{6}$$

Exercise 3. Let $\|\cdot\|$ be any norm on $\mathbb{R}^N$ (that is satisfy the three properties). Define $A := \{\boldsymbol{x} \in \mathbb{R}^N |\, \|\boldsymbol{x}\| < 1\}$. Prove that A is convex.

2. Solutions to exercises

Exercise 1. (A, B) is an inner product. The norm is the Frobenius norm:

$$\|A\|_F := \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}. \tag{7}$$

Exercise 2. We have

$$\begin{aligned} \|A\,\boldsymbol{x}\| &= [(a_{11}\,x_1 + \cdots + a_{1N}\,x_N)^2 + \cdots]^{1/2} \\ &\leqslant [(a_{11}^2 + \cdots + a_{1N}^2)\,(x_1^2 + \cdots + x_N^2) + \cdots] \\ &= \left(\sum_{i,j} a_{ij}^2 \right) (x_1^2 + \cdots + x_N^2) \\ &= \|A\|_F\, \|\boldsymbol{x}\|. \end{aligned} \tag{8}$$

Note that we have used Cauchy-Schwarz in the inequality step.

Exercise 3. For any $\boldsymbol{x}, \boldsymbol{y} \in A$ and $t \in [0, 1]$, we have

$$\begin{aligned} \|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\| &\leqslant \|t\,\boldsymbol{x}\| + \|(1-t)\,\boldsymbol{y}\| \\ &= |t|\,\|\boldsymbol{x}\| + |1-t|\,\|\boldsymbol{y}\| \\ &= t\,\|\boldsymbol{x}\| + (1-t)\,\|\boldsymbol{y}\| \\ &< t + (1-t) = 1. \end{aligned} \tag{9}$$

3. Problems

Problem 1. Let $A \subseteq \mathbb{R}^N$ be convex. Prove that $A^o, \bar{A}$ are convex.

B. Topology of $\mathbb{R}^N$

1. Exercises

Exercise 4. Let $A := \{(x, y) | \, x\, y > 2\}$. Prove that A is open.

Exercise 5. Let $A \subseteq \mathbb{R}^N$ be defined through

$$x_1 + x_2 = 1, \qquad x_1^2 + x_2^2 < 1. \tag{10}$$

Is A open or closed or both or neither? Justify your answer.

Exercise 6. Let $A, B \subseteq \mathbb{R}^N$. Prove $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

Exercise 7. Let $A \subseteq \mathbb{R}^N$ be compact. Let W be a collection of closed sets satisfying $A \cap (\cap_{E \in W} E) = \varnothing$. Prove that there are E_1, ..., $E_n \in W$ such that $A \cap (\cap_{k=1}^n E_k) = \varnothing$.

2. Solutions to exercise

Exercise 4. Take any $(x_0, y_0) \in A$. We need to find $r > 0$ such that $B((x_0, y_0), r) \subseteq A$. Denote $m := x\, y - 2 > 0$. Now take

$$r := \min \left\{ 1, \frac{m}{|x_0| + |y_0| + 1} \right\}. \tag{11}$$

Then for any $(x, y) \in B((x_0, y_0), r)$

we have

$$\begin{aligned} x\, y &= x_0\, y_0 + x_0\, u + y_0\, v + u\, v \\ &\geqslant x_0\, y_0 - |x_0\, u| - |y_0\, v| - |u|\, |v| \\ &\geqslant x_0\, y_0 - [|x_0| + |y_0|]\, r - r^2 \\ &> 2 + m - [|x_0| + |y_0| + 1]\, r \\ &\geqslant 2. \end{aligned} \tag{12}$$

Exercise 5. The set is neither open nor closed.

- Not open. Take any $\boldsymbol{x} \in A$ and any $r > 0$. Then

$$\boldsymbol{x}' := \boldsymbol{x} + \frac{r}{2} \boldsymbol{e}_1 + \frac{r}{2} \boldsymbol{e}_2 \in B(\boldsymbol{x}, r) \tag{13}$$

  but

$$x_1' + x_2' = x_1 + x_2 + r = 1 + r \neq 1 \tag{14}$$

  so $\boldsymbol{x}' \notin A$.

- Not closed. We prove A^c is not open. Clearly $\boldsymbol{e}_1 \notin A$. Now for any $r > 0$, define

$$r' := \min \{r, 1\}. \tag{15}$$

  Consider

$$\boldsymbol{x} := \left(1 - \frac{r'}{2} \right) \boldsymbol{e}_1 - \frac{r'}{2} \boldsymbol{e}_2. \tag{16}$$

  Then clearly $\boldsymbol{x} \in B(\boldsymbol{e}_1, r) \cap A$ so A^c is not open.

Exercise 6. Since $A \subseteq \bar{A}$, $B \subseteq \bar{B}$, $A \cup B \subseteq \bar{A} \cup \bar{B}$. As the latter is closed, we have

$$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}. \tag{17}$$

For the other direction, as $A \subseteq A \cup B$, we have $\bar{A} \subseteq \overline{A \cup B}$. Similarly $\bar{B} \subseteq \overline{A \cup B}$. Therefore

$$\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}. \tag{18}$$

Exercise 7. Sincer $A \cap (\cap_{E \in W} E) = \varnothing$, we have

$$A \subseteq \cup_{E \in W} E^c. \tag{19}$$

This is an open covering of the compact set A so there is a finite sub-cover:

$$A \subseteq E_1^c \cup \cdots \cup E_n^c. \tag{20}$$

Consequently

$$A \cap (E_1 \cap \cdots \cap E_n) = \varnothing. \tag{21}$$

3. Problems

Problem 2. Let $A := \left\{ \left(x, \sin \frac{1}{x} \right) | \, x, y \in \mathbb{R}, x, y \neq 0 \right\}$. Find

- A^o;
- $\bar{A}$;
- ∂A;
- Cluster points of A.

Problem 3. Let $A \subseteq \mathbb{R}^N$. Prove: $\partial(\partial A) \subseteq \partial A$. Then find counter-examples for the following claims:

- $\partial(\partial A) \subset \partial A$ (meaning: $\subseteq$ but not $=$)
- $\partial(\partial A) = \partial A$.

Problem 4. Let $A, B \subseteq \mathbb{R}^N$ with A open and B compact. Prove that there is an open set $V \subseteq \mathbb{R}^N$ such that

$$B \subseteq V, \qquad \bar{V} \subseteq A. \tag{22}$$

C. Continuity of Functions

1. Exercises

Exercise 8. Prove that

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{\sin(x\,y)}{x^2+y^2} \tag{23}$$

does not exist.

Exercise 9. Let $f(x,y) := \begin{cases} \exp\left(-\frac{1}{|x|+|y|}\right) & (x,y)\neq \mathbf{0} \\ 0 & (x,y)=\mathbf{0} \end{cases}$. Prove that f is continuous at $(0,0)$.

Exercise 10. Prove that if the limit $\lim_{(x,y)\longrightarrow(0,0)} f(x) + g(y)$ exists, then the limits $\lim_{x\longrightarrow 0} f(x)$ and $\lim_{y\longrightarrow 0} g(y)$ both exist.

2. Solutions to exercises

Exercise 8. Denote $f(x,y) := \frac{\sin(x\,y)}{x^2+y^2}$. For any $r>0$, we have $\left(\frac{r}{2},0\right)\in B(\mathbf{0},r)$ and $f\left(\frac{r}{2},0\right)=0$.

On the other hand, since $\lim_{x\longrightarrow 0} \frac{\sin x}{x} = 1$, there is $\delta>0$ such that for all $0<|x|<\delta^2$,

$$\frac{\sin x}{x} > \frac{1}{2}. \tag{24}$$

Now consider $\delta' := \min\{\delta, r\}$ and set $(x,y) = (\delta'/2, \delta'/2)\in B(\mathbf{0},r)$. Then

$$f(\delta'/2, \delta'/2) = \frac{\sin((\delta')^2)}{2\,(\delta')^2} > \frac{1}{4}. \tag{25}$$

Thus the limit cannot exist.

Exercise 9. For any $\varepsilon>0$, take $\delta < (-\ln\varepsilon)^{-1}/2$. Then for all (x,y) such that $\|(x,y)\|<\delta$, we have

$$|x|+|y| \leqslant 2\,(x^2+y^2)^{1/2} < 2\,\delta. \tag{26}$$

Now we have

$$\left|e^{-1/(|x|+|y|)} - 0\right| < \varepsilon. \tag{27}$$

Exercise 10. For any $\varepsilon > 0$, since $\text{im}_{(x,y)\longrightarrow(0,0)} f(x) + g(y)$ exists, there is $\delta>0$ such that for all $(x_1,y_1),(x_2,y_2)\in B(\mathbf{0},\delta)$,

$$|[f(x_1)+g(y_1)] - [f(x_2)+g(y_2)]| < \varepsilon. \tag{28}$$

Now for any x_1, x_2 such that $|x_1|, |x_2| < \delta$, we have

$$(x_1,0),(x_2,0)\in B(\mathbf{0},\delta) \tag{29}$$

which gives

$$|f(x_1) - f(x_2)| < \varepsilon. \tag{30}$$

Therefore $\lim_{x\longrightarrow 0} f(x)$ exists. Similarly $\lim_{y\longrightarrow 0} g(y)$ exists.

3. Problems

Problem 5. Let $f\colon \mathbb{R}^N \mapsto \mathbb{R}$ be continuous. Denote

$$[f<0] := \{\boldsymbol{x}\in\mathbb{R}^N |\ f(\boldsymbol{x})<0\} \tag{31}$$

and

$$[f=0] := \{\boldsymbol{x}\in\mathbb{R}^N |\ f(\boldsymbol{x})=0\}. \tag{32}$$

Prove that $\partial[f<0]\subset[f=0]$. Does equality hold? What if we take away the continuity assumption?

Problem 6. Let $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$ be bounded and continuous. Prove that $\boldsymbol{f}$ is continuous if and only if its graph $\{(\boldsymbol{x},\boldsymbol{y})|\ \boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})\}$ is a closed set in $\mathbb{R}^{N+M}$. Then discuss:

- What if we remove the boundedness assumption?

D. Differentiability of Functions

1. Exercises

Exercise 11. Let $f(x,y)=x\,y\sin\left(\frac{1}{x^2+y^2}\right)$ for $(x,y)\neq(0,0)$ and $f(0,0)=0$. Prove that f is differentiable at $(0,0)$ and find its differential there.

Exercise 12. Calculate partial derivatives for $f(x,y,z)=\sin(x\,y\,z)$.

Exercise 13. Prove that $f(x,y)=e^{x\,y}$ is differentiable.

Exercise 14. Let $f(x,y)$ be differentiable. Define

$$u(r,\theta):=f(r\cos\theta,r\sin\theta) \tag{33}$$

Prove

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2. \tag{34}$$

Here the left hand side is evaluated at $(x,\,y)=(r\cos\theta,\,r\sin\theta)$.

2. Solutions to exercises

Exercise 11.

We prove $Df(0,0)=0$. That is for any $(x,y)\in\mathbb{R}^2$,

$$[Df(0,0)](x,y)=0. \tag{35}$$

To do this we check

$$\left|x\,y\sin\frac{1}{x^2+y^2}\right|\leqslant|x\,y|\leqslant(x^2+y^2) \tag{36}$$

therefore

$$\lim_{(x,y)\longrightarrow(0,0)}\frac{\left|x\,y\sin\frac{1}{x^2+y^2}-0\right|}{(x^2+y^2)^{1/2}}=0. \tag{37}$$

Exercise 12.

$$\frac{\partial f}{\partial x} = y\,z\cos(x\,y\,z); \tag{38}$$

$$\frac{\partial f}{\partial y} = x\,z\cos(x\,y\,z); \tag{39}$$

$$\frac{\partial f}{\partial z} = x\,y\cos(x\,y\,z). \tag{40}$$

Exercise 13. We calculate

$$\frac{\partial f}{\partial x}=y\,e^{x\,y},\qquad \frac{\partial f}{\partial y}=x\,e^{x\,y}. \tag{41}$$

Both are continuous at all $(x,\,y)\in\mathbb{R}^2$. Therefore f is differentiable at every $(x,\,y)\in\mathbb{R}^2$.

Exercise 14. We calculate through chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta; \tag{42}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta)+\frac{\partial f}{\partial y}(r\cos\theta). \tag{43}$$

Now clearly the conclusion holds.

3. Problems

Problem 7. Let $f(x,\,y)$: $\mathbb{R}^2\mapsto\mathbb{R}$. Assume $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exists at all $(x,\,y)\in\mathbb{R}^2$. Prove that

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \text{ for all } (x,\,y)\Longleftrightarrow f \text{ is constant.} \tag{44}$$

Problem 8. Let $\boldsymbol{f}$: $\mathbb{R}^N\mapsto\mathbb{R}^M$. Assume all its partial derivatives are bounded, that is there is $K>0$ such that

$$\forall\boldsymbol{x}\in\mathbb{R}^N,\forall i=1,\ldots,M,\,j=1,\ldots,N\quad\left|\frac{\partial f_i}{\partial x_j}(\boldsymbol{x})\right|\leqslant K. \tag{45}$$

Prove that $\boldsymbol{f}$ is uniformly continuous.

Problem 9. Let u,v be differentiable and satisfy

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\ u^2+v^2=R^2 \tag{46}$$

for some constant R. Prove that both u,v are constants.

Problem 10. Let $f,\,g$: $\mathbb{R}^N\mapsto\mathbb{R}$ and $\boldsymbol{x}_0\in\mathbb{R}^N$. Assume f is continuous at $\boldsymbol{x}_0$ and g is differentiable there with $g(\boldsymbol{x}_0)=0$. Prove that $f\,g$ is differentiable with differential $f(\boldsymbol{x}_0)\,Dg(\boldsymbol{x}_0)$.

E. Implicit and Inverse Functions

1. Exericises

Exercise 15. Let $y = Y(x)$ be defined through the implicit relation

$$x^2 + 2\,x\,y - y^2 = a^2. \tag{47}$$

Calculate Y', Y''.

Exercise 16. Let $z = Z(x, y)$ be defined through

$$x + y + z = e^{x+y+z}. \tag{48}$$

Calculate $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$.

Exercise 17. Let the implicit functions $r = R(x,\ y)$, $\theta = \Theta(x, y)$ be define through

$$\begin{aligned} x &= r\cos\theta & (49)\\ y &= r\sin\theta. & (50) \end{aligned}$$

Find $\frac{\partial(R, \Theta)}{\partial(x, y)}$.

2. Solutions to exercises

Exercise 15. We have

$$x^2 + 2\,x\,Y(x) - Y(x)^2 = a^2. \tag{51}$$

Taking derivative:

$$2\,x + 2\,Y + 2\,x\,Y' - 2\,Y Y' = 0. \tag{52}$$

which gives

$$Y'(x) = \frac{x + y}{x - y}. \tag{53}$$

Taking derivative one more time:

$$2 + 4\,Y' + 2\,x\,Y'' - 2\,(Y')^2 - 2\,Y Y'' = 0 \tag{54}$$

This gives

$$Y'' = \frac{(Y')^2 - 2\,Y' - 1}{x - y} \tag{55}$$

which simplifies to

$$Y'' = \frac{(x + y)\,(3\,y - x) - (x - y)^2}{(x - y)^3}. \tag{56}$$

Exercise 16. $Z(x, y)$ satisfies

$$x + y + Z = e^{x+y+Z}. \tag{57}$$

Taking $\frac{\partial}{\partial x}$ we have

$$1 + \frac{\partial Z}{\partial x} = e^{x+y+Z}\left(1 + \frac{\partial Z}{\partial x}\right) \tag{58}$$

which gives either $x + y + Z = 0$ which is not possible, or $\frac{\partial Z}{\partial x} = -1$. Similarly we have $\frac{\partial Z}{\partial y} = -1$.

Exercise 17. We have

$$I = \frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r, \theta)}\,\frac{\partial(R, \Theta)}{\partial(x, y)} \tag{59}$$

which gives

$$\begin{aligned} \frac{\partial(R, \Theta)}{\partial(x, y)} &= \left[r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right]^{-1} \\ &= \frac{1}{r}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{60}$$

3. Problems

Problem 11. Let $z = Z(x, y)$ be defined through

$$x^2 + y^2 + z^2 = y\,f\left(\frac{z}{y}\right) \tag{61}$$

for some differentiable function f. Prove that Z satisfy the following partial differential equation:

$$(x^2 - y^2 - z^2)\,\frac{\partial Z}{\partial x} + 2\,x\,y\,\frac{\partial Z}{\partial y} = 2\,x\,Z. \tag{62}$$

Solutions to Problems

Problem 1.

- A^o.

  Take any $\boldsymbol{x}$, $\boldsymbol{y} \in A^o$ and $t \in (0,1)$. Denote $\boldsymbol{x}_t := t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}$. All we need to show is that there is $r > 0$ such that $B(\boldsymbol{x}_t, r) \subseteq A$.

  Since $\boldsymbol{x}$, $\boldsymbol{y} \in A^o$, there is r_x, $r_y > 0$ such that $B(\boldsymbol{x}, r_x) \subseteq A$, $B(\boldsymbol{y}, r_y) \subseteq A$. Now take $r = \min\{r_x, r_y\}$ and we claim that $B(\boldsymbol{x}_t, r) \subseteq A$ for all $\boldsymbol{x}_t$.

  Take $\boldsymbol{z}_t \in B(\boldsymbol{x}_t, r)$. Define

  $$\boldsymbol{z}_x := \boldsymbol{z}_t + (\boldsymbol{x} - \boldsymbol{x}_t);\ \boldsymbol{z}_y := \boldsymbol{z}_t + (\boldsymbol{y} - \boldsymbol{x}_t). \tag{63}$$

  Then we have

  $$\boldsymbol{z}_t = t\,\boldsymbol{z}_x + (1-t)\,\boldsymbol{z}_y. \tag{64}$$

  Now check

  $$\|\boldsymbol{x} - \boldsymbol{z}_x\| = \|\boldsymbol{z}_t - \boldsymbol{x}_t\| < r \leqslant r_x \tag{65}$$

  and similarly $\|\boldsymbol{y} - \boldsymbol{z}_y\| < r_y$. Therefore

  $$\boldsymbol{z}_x \in B(\boldsymbol{x}, r_x) \subseteq A,\ \boldsymbol{z}_y \in B(\boldsymbol{y}, r_y) \subseteq A. \tag{66}$$

  By convexity of A we have $\boldsymbol{z}_t \in A$. The arbitrariness of $\boldsymbol{z}_t$ now yields $B(\boldsymbol{x}_t, r) \subseteq A$ and consequently $\boldsymbol{x}_t \in A^o$.

- $\bar{A}$.

  Take any $\boldsymbol{x}$, $\boldsymbol{y} \in \bar{A}$ and any $t \in (0,1)$. We need to prove $\boldsymbol{x}_t := t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in \bar{A}$. Note that it suffices to prove for any $r > 0$,

  $$B(\boldsymbol{x}_t, r) \cap A \neq \varnothing. \tag{67}$$

  Now take any $r > 0$. Since $\boldsymbol{x}, \boldsymbol{y} \in \bar{A}$, there are $\boldsymbol{x}' \in B(\boldsymbol{x}, r) \cap A$, $\boldsymbol{y}' \in B(\boldsymbol{y}, r) \cap A$. Therefore

  $$\boldsymbol{x}_t' := t\,\boldsymbol{x}' + (1-t)\,\boldsymbol{y}' \in A. \tag{68}$$

  Now we calcualte

  $$\begin{aligned} \|\boldsymbol{x}_t - \boldsymbol{x}_t'\| &= \|t\,(\boldsymbol{x} - \boldsymbol{x}') + (1-t)\,(\boldsymbol{y} - \boldsymbol{y}')\| \\ &< r. \end{aligned} \tag{69}$$

  Thus $B(\boldsymbol{x}_t, r) \cap A \neq \varnothing$ and the proof ends.

Problem 2. Let $A := \left\{\left(x, \sin\frac{1}{x}\right) | \, x, y \in \mathbb{R}, x, y \neq 0\right\}$. Find

- $A^o = \varnothing$.

  Take any $(x_0, y_0) \in A$ then we have

  $$y_0 = \sin\frac{1}{x_0}. \tag{70}$$

  For any $r > 0$, clearly

  $$(x_0, y_0 + r/2) \in B((x_0, y_0), r) \tag{71}$$

  but does not belong to A.

- $\bar{A} = A \cup B$ with $B := \{(0, y) | \, y \in [-1, 1]\}$.

  We first prove that $A \cup B$ is closed, then prove that for any $y_0 \in [-1, 1]$ and any $r > 0$, $B((0, y_0), r) \cap A \neq \varnothing$.

  - $A \cup B$ is closed. We prove its complement is open. Take any $(x_0, y_0) \notin A \cup B$.
    - Case 1. $x_0 = 0$. Then $y_0 > 1$. Take $r := y_0 - 1$. Clearly $B((x_0, y_0), r) \cap (A \cup B) = \varnothing$.
    - Case 2. $x_0 \neq 0$. Then $y_0 \neq \sin\left(\frac{1}{x_0}\right)$. Set

      $$\varepsilon_0 := \left| y_0 - \sin\left(\frac{1}{x_0}\right) \right| > 0. \tag{72}$$

      Now since $\sin\left(\frac{1}{x}\right)$ is continuous at x_0, there is $\delta_0 > 0$ such that for all $|x - x_0| < \delta_0$, $\left|\sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x_0}\right)\right| < \varepsilon_0/2$. Now set $r: = \min\{\delta_0, \varepsilon_0/2, |x_0|\}$. Then for any $(x, y) \in B((x_0, y_0), r)$, we have

      $$\left| y_0 - \sin\left(\frac{1}{x}\right) \right| > \frac{\varepsilon_0}{2} \tag{73}$$

      while

      $$|y - y_0| < \frac{\varepsilon_0}{2}, \qquad |x - x_0| < |x_0|. \tag{74}$$

      Consequently $(x, y) \notin A \cup B$. Thus

      $$B((x_0, y_0), r) \cap (A \cup B) = \varnothing. \tag{75}$$

  - $A \cup B$ is the smallest closed set containing A. To show this it is enough to prove that for any $(0, y_0) \in B$ and any $r > 0$, $B((0, y_0), r) \cap A \neq \varnothing$.

    Take an arbitrary $(0, y_0) \in B$ and any $r > 0$. Then there is $n \in \mathbb{N}$ such that $2\,n\,\pi > r^{-1}$. This gives

    $$\left|\frac{1}{2\,n\,\pi + \pi/2}\right|, \ \left|\frac{1}{2\,n\,\pi + 3\,\pi/2}\right| < r. \tag{76}$$

    But we have

    $$\sin(2\,n\,\pi + \pi/2) = 1, \tag{77}$$

    $$\sin(2\,n\,\pi + 3\,\pi/2) = -1. \tag{78}$$

Thus by intermediate value theorem, there is $x \in \left(\frac{1}{2n\pi + 3\pi/2}, \frac{1}{2n\pi + \pi/2} \right)$ such that $\sin\left(\frac{1}{x}\right) = y_0$. Now we have

$$\left(x, \sin\left(\frac{1}{x}\right) \right) \in A \cap B((0, y_0), r). \tag{79}$$

Thus ends the proof.

- $\partial A = B := \{(0, y) |\ y \in [-1, 1]\}$.
- Cluster points of A is the same as $\bar{A}$.
  - For any $(x_0,\ y_0) \in A$, since $\sin \frac{1}{x}$ is continuous at x_0, taking any $x_n \longrightarrow x_0$, $x_n \neq x_0$ we have $\sin \frac{1}{x_n} \longrightarrow \sin \frac{1}{x_0}$. Consequently for any $r > 0$, there is $n \in \mathbb{N}$ such that

    $$\left(x_n,\ \sin \frac{1}{x_n} \right) \in A \cap B((x_0,\ y_0)) - \{(x_0, y_0)\}. \tag{80}$$

  - For any $(0, y_0) \in B$, similar to the proof of $A \cup B$ is smallest closed set containing A, we can find $x_n \neq 0$, $x_n \longrightarrow 0$ with $\sin\left(\frac{1}{x_n}\right) = y_0$.

Problem 3.

- Proof of $\partial(\partial A) \subseteq \partial A$. We first prove ∂A is closed. This follows immediately from the definition:

  $$\partial A = \bar{A} - A^o = \bar{A} \cap (A^o)^c. \tag{81}$$

  Now

  $$\partial(\partial A) = \overline{\partial A} - (\partial A)^o = \partial A - (\partial A)^o \subseteq \partial A. \tag{82}$$

- Counter-example for $\partial(\partial A) \subset \partial A$. Take $A = \{\boldsymbol{x}_0\}$. Then $\bar{A} = A$, $A^o = \varnothing$. So $\partial A = A$. Then of course $\partial(\partial A) = \partial A$.
- Counter-example for $\partial(\partial A) = \partial A$. Take $A = \mathbb{Q} \subset \mathbb{R}$. Then $A^o = \varnothing$, $\bar{A} = \mathbb{R}$ so $\partial A = \mathbb{R}$. Now $(\partial A)^o = \overline{\partial A} = \mathbb{R}$ so $\partial(\partial A) = \varnothing$.

Problem 4. For any $\boldsymbol{x} \in B$, there is $r_x > 0$ such that

$$B(\boldsymbol{x}, r_x) \subseteq A. \tag{83}$$

Now consider the open covering of B:

$$B \subseteq \cup_{\boldsymbol{x} \in B} B(\boldsymbol{x}, r_x/2). \tag{84}$$

There is a finite sub-covering:

$$B \subseteq B(\boldsymbol{x}_1, r_1/2) \cup \cdots \cup B(\boldsymbol{x}_n, r_n/2). \tag{85}$$

Now define

$$V := B(\boldsymbol{x}_1, r_1/2) \cup \cdots \cup B(\boldsymbol{x}_n, r_n/2). \tag{86}$$

Clearly V is open and $B \subseteq V$.

Now we prove $\bar{V} \subseteq A$. We have

$$\begin{aligned} \bar{V} &= \overline{B(\boldsymbol{x}_1, r_1/2)} \cup \cdots \cup \overline{B(\boldsymbol{x}_n, r_n/2)} \\ &\subseteq B(\boldsymbol{x}_1, r_1) \cup \cdots \cup B(\boldsymbol{x}_n, r_n) \\ &\subseteq A. \end{aligned} \tag{87}$$

Problem 5.

- Proof of $\partial[f < 0] \subset [f = 0]$. Take any $\boldsymbol{x}_0 \in \partial[f < 0]$. Then for any $r > 0$,

  $$B(\boldsymbol{x}_0, r) \cap [f < 0] \neq \varnothing, \tag{88}$$

  $$B(\boldsymbol{x}_0, r) \cap [f \geqslant 0] \neq \varnothing. \tag{89}$$

  Now we proceed via proof by contradiction. Assume $f(\boldsymbol{x}_0) \neq 0$. Consider two cases.

  - Case 1. $f(\boldsymbol{x}_0) > 0$. Then there is $r > 0$ such that for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$, $|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < |f(\boldsymbol{x}_0)| \Longrightarrow f(\boldsymbol{x}) > 0$.
  - Case 2. $f(\boldsymbol{x}_0) < 0$. Then there is $r > 0$ such that for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$, $|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < |f(\boldsymbol{x}_0)| \Longrightarrow f(\boldsymbol{x}) < 0$.

  Either way we contradicts one of (88–89).

- Equality may not hold. For example take $f(\boldsymbol{x}) = 0$. Then $[f < 0] = \varnothing$ and consequently $\partial[f < 0] = \varnothing$. But $[f = 0] = \mathbb{R}^N$.
- If continuity assumption is dropped, the conclusion does not hold. For example take $f(x) = \begin{cases} -1 & x \neq 0 \\ 1 & x = 0 \end{cases}$. Then $\partial[f < 0] = \{0\}$ but $[f = 0] = \varnothing$.

Problem 6.

- Proof of "$\boldsymbol{f}$ is continuous if and only if its graph $\{(\boldsymbol{x},\ \boldsymbol{y}) |\ \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\}$ is a closed set in $\mathbb{R}^{N+M}$". Denote the graph by $G \subseteq \mathbb{R}^{N+M}$.
  - If. We prove by contradiction. Assume $\boldsymbol{f}$ is not continuous. Then there is $\varepsilon_0 > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^N$ such that there is $\boldsymbol{x}_n \longrightarrow \boldsymbol{x}_0$ with $\|\boldsymbol{f}(\boldsymbol{x}_n) - \boldsymbol{f}(\boldsymbol{x}_0)\| > \varepsilon_0$. Now since $\boldsymbol{f}(\boldsymbol{x}_n)$ is bounded, there is a convergent subsequence

    $$\boldsymbol{f}(\boldsymbol{x}_{n_k}) \longrightarrow \boldsymbol{L}. \tag{90}$$

Necessarily $\boldsymbol{L} \neq \boldsymbol{f}(\boldsymbol{x}_0)$. But now we have

$$(\boldsymbol{x}_{n_k}, \boldsymbol{f}(\boldsymbol{x}_{n_k})) \longrightarrow (\boldsymbol{x}_0, \boldsymbol{L}) \tag{91}$$

which is not in graph of $\boldsymbol{f}$. Contradiction.

– Only if. Still prove by contradiction. Assume there is $(\boldsymbol{x}_0, \boldsymbol{y}_0) \notin G$ such that for any $r > 0$, $B((\boldsymbol{x}_0, \boldsymbol{y}_0), r) \cap G \neq \varnothing$. Then we can find $\boldsymbol{x}_n \longrightarrow \boldsymbol{x}_0$ such that $\boldsymbol{f}(\boldsymbol{x}_n) \longrightarrow \boldsymbol{y}_0$. But then by continuity of $\boldsymbol{f}$ it must hold that $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$. Contradiction.

- What if we remove the boundedness assumption?

  The conclusion does not hold anymore. For example $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then its graph is closed but f is not continuous.

Problem 7.

- $\Longrightarrow$. Fix $(x_0, y_0) \in \mathbb{R}^2$. For any $(x, y) \in \mathbb{R}^2$, by Mean Value Theorem, there are ξ_1, ξ_2 such that

$$\begin{aligned} f(x, y) - f(x_0, y_0) &= \frac{\partial f}{\partial x}(\xi_1, y_0)\,(x - x_0) \\ &\quad + \frac{\partial f}{\partial y}(x, \xi_2)\,(y - y_0) \\ &= 0. \end{aligned} \tag{92}$$

  Therefore f is a constant.

- $\Longleftarrow$. This direction is obvious.

Problem 8.

Take any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. By MVT we have

$$\begin{aligned} |f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| &\leqslant K \sum_{j=1}^{N} |x_j - y_j| \\ &\leqslant K \sum_{j=1}^{N} \|\boldsymbol{x} - \boldsymbol{y}\| \\ &= K N \, \|\boldsymbol{x} - \boldsymbol{y}\|. \end{aligned} \tag{93}$$

This gives

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| &= \left[\sum_{i=1}^{M} (f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}))^2\right]^{1/2} \\ &\leqslant \left[\sum_{i=1}^{M} (K N \, \|\boldsymbol{x} - \boldsymbol{y}\|)^2\right]^{1/2} \\ &= \sqrt{M}\, N K \, \|\boldsymbol{x} - \boldsymbol{y}\|. \end{aligned} \tag{94}$$

Now for any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{\sqrt{M}\, N K}$. We have whenever $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$, $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \varepsilon$. So it is uniformly continuous.

Problem 9. Since $u^2 + v^2 = R^2$ we consider two cases.

- Case 1. $R = 0$. Then clearly $u = v = 0$.
- Case 2. $R \neq 0$. We have

$$\begin{aligned} 0 &= \frac{\partial(u^2 + v^2)}{\partial x} \\ &= 2\, u \frac{\partial u}{\partial x} + 2\, v \frac{\partial v}{\partial x} \\ &= 2 \left[u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} \right]; \end{aligned} \tag{95}$$

$$0 = \frac{\partial(u^2 + v^2)}{\partial y} = 2 \left[u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial x} \right]. \tag{96}$$

  Thus $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ satisfy

$$u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} = 0 \tag{97}$$

$$v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0. \tag{98}$$

  Solving this system we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \tag{99}$$

  which leads to u being constant. The proof for v being constant is similar and omitted.

Problem 10. Denote $D g(\boldsymbol{x}_0)$ by L.

First assume $f(\boldsymbol{x}_0) \neq 0$. For any $\varepsilon > 0$, take $\delta > 0$ such that for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, \delta)$,

$$|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < \frac{\varepsilon}{2\,(L + 1)} \tag{100}$$

$$\frac{|\boldsymbol{g}(\boldsymbol{x}) - L(\boldsymbol{x} - \boldsymbol{x}_0)|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} < \min\left\{\frac{\varepsilon}{2\,|f(\boldsymbol{x}_0)|}, 1\right\} \tag{101}$$

Then we have, for all such $\boldsymbol{x}$,

$$\begin{aligned} & \frac{|(f g)(\boldsymbol{x}) - (f g)(\boldsymbol{x}_0) - f(\boldsymbol{x}_0)\, L(\boldsymbol{x} - \boldsymbol{x}_0)|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \\ = & \frac{|f(\boldsymbol{x})\, g(\boldsymbol{x}) - f(\boldsymbol{x}_0)\, L(\boldsymbol{x} - \boldsymbol{x}_0)|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \\ \leqslant & \frac{|f(\boldsymbol{x}_0)\, [g(\boldsymbol{x}) - L(\boldsymbol{x} - \boldsymbol{x}_0)]|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \\ & + \frac{|[f(\boldsymbol{x}) - f(\boldsymbol{x}_0)]\, g(\boldsymbol{x})|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{102}$$

The claim is proved.

In the case $f(\boldsymbol{x}_0)=0$ we can simply choose δ such that

$$\frac{|\boldsymbol{g}(\boldsymbol{x})-L(\boldsymbol{x}-\boldsymbol{x}_0)|}{\|\boldsymbol{x}-\boldsymbol{x}_0\|}<1. \qquad (103)$$

Problem 11. Taking $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of $x^2+y^2+z^2=y\,f\left(\frac{z}{y}\right)$ we have (all f' are evaluated at Z/y):

$$2\,x+2\,Z\,\frac{\partial Z}{\partial x} = f'\,\frac{\partial Z}{\partial x} \qquad (104)$$

$$2\,y+2\,Z\,\frac{\partial Z}{\partial y} = f+f'\,\frac{\partial Z}{\partial y}-f'\,\frac{Z}{y}. \qquad (105)$$

This gives

$$\frac{\partial Z}{\partial x}=\frac{2\,x}{f'-2\,Z} \qquad (106)$$

and

$$\frac{\partial Z}{\partial y}=\frac{2\,y-f+f'\,Z/y}{f'-2\,Z}. \qquad (107)$$

Now we calculate

$$\begin{aligned}
&(x^2-y^2-z^2)\,\frac{\partial Z}{\partial x}+2\,x\,y\,\frac{\partial Z}{\partial y}\\
=\;&\frac{2\,x\,(x^2-y^2-z^2)}{f'-2\,Z}+\frac{4\,x\,y^2-2\,x\,y\,f+2\,x\,f'\,Z}{f'-2\,z}\\
=\;&\frac{2\,x\,(x^2+y^2-z^2)-2\,x\,(x^2+y^2+z^2)+2\,x\,f'\,Z}{f'-2\,Z}\\
=\;&\frac{-4\,x\,Z^2+2\,x\,f'\,Z}{f'-2\,Z}=2\,x\,Z. \qquad (108)
\end{aligned}$$