Math 217 Fall 2013 Homework 9 Solutions

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- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Consider the following set:*

$$A := \left\{ \left(\frac{p}{q}, \frac{r}{q} \right) | \, p, q \in \mathbb{N}, (p, q), (r, q) \text{ co-prime} \right\} \cap [0, 1]^2. \tag{1}$$

Prove that for every $\alpha \in [0, 1]$, $\mu(A \cap \{x = \alpha\}) = \mu(A \cap \{y = \alpha\}) = 0$, *therefore*

$$\int_0^1 \mu(A \cap \{x = t\}) \, \mathrm{d}t = \int_0^1 \mu(A \cap \{y = t\}) \, \mathrm{d}t = 0, \tag{2}$$

but $\mu(A)$ *does not exist. (This example is constructed by A. Pringsheim in 1898).*

Question 2. *Calculate*

$$\int_D x^2 \, y^2 \, \mathrm{d}(x, y) \tag{3}$$

where D *is the triangle enclosed by* $y = \frac{b}{a} x, y = 0, x = a$.

Question 3. *Calculate*

$$\int_D (x^2 + y^2) \, \mathrm{d}(x, y) \tag{4}$$

where D *is enclosed by*

$$y = a + x, y = x, y = a, y = 3\, a. \tag{5}$$

Question 4. *Calculate the area enclosed by* $y = x^2$ *and* $y^2 = x$.

Question 5. *Let* $f(x, y)$ *be continuous on* $I := [a, b] \times [c, d]$. *Define for* $(x, y) \in I$,

$$F(x, y) := \int_{[a,x] \times [c,y]} f(u, v) \, \mathrm{d}(u, v). \tag{6}$$

Prove that

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x} = f(x,y). \tag{7}$$

Question 6. *Let* $I := [a,b] \times [c,d]$*. Let* $f(x) \colon [a,b] \mapsto \mathbb{R}$, $g(x) \colon [c,d] \mapsto \mathbb{R}$*. Let* $F(x,y) := f(x)\, g(y)$*.*

a) *Prove that* $F(x,\ y)$ *is integrable on* I *if* $f,\ g$ *are integrable on* $[a,\ b]$, $[c,\ d]$ *respectively. Furthermore we have*

$$\int_I F(x,y)\,\mathrm{d}(x,y) = \left[\int_a^b f(x)\,\mathrm{d}x\right]\left[\int_c^d g(x)\,\mathrm{d}x\right]. \tag{8}$$

b) *Does it hold that* $F(x,y)$ *is integrable only if* f, g *are integrable?*

c) *Prove*

$$\left[\int_a^b f(x)\,\mathrm{d}x\right]^2 \leqslant (b-a)\int_a^b f(x)^2\,\mathrm{d}x \tag{9}$$

though studying

$$\int_{[a,b]^2} [f(x) - f(y)]^2\,\mathrm{d}(x,y). \tag{10}$$