Math 217 Fall 2013 Homework 8 Solutions

Due Thursday Nov. 14, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $A := \{(x,y) \in [0,1]^2 | \, x \in \mathbb{Q}, y \notin \mathbb{Q}\}$. Is A Jordan measurable? Justify.*

Solution. Since both $\mathbb{Q} \cap [0,1]$ and $[0,1] - \mathbb{Q}$ are dense in $[0,1]$, A is dense in $[0,1]^2$. Therefore $\bar{A} = [0,1]^2$ and $\mu_{\text{out}}(A) = 1$.

On the other hand, $A^c = \{(x, \, y) \in [0, \, 1]^2 | \; x \notin \mathbb{Q}, \, y \in \mathbb{Q}\}$ is also dense in $[0,1]^2$ so $A^o = \varnothing$. Consequently $\mu_{\text{in}}(A) = 0$.

So A is not Jordan measurable.

Question 2. *Let $A := \left\{ \left(\frac{1}{m}, \frac{1}{n} \right) | \, m, n \in \mathbb{N} \right\}$. Prove that $\mu(A) = 0$.*

Solution. We have

$$\bar{A} \subseteq A \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}). \tag{1}$$

Now for any $\varepsilon > 0$, define

$$I := \left[0, \frac{\varepsilon}{4}\right] \times [0,1]; \qquad J := [0,1] \times \left[0, \frac{\varepsilon}{4}\right]. \tag{2}$$

Then

$$\bar{A} \subseteq Q := (I \cup J) \cup \left\{ \left(\frac{1}{m}, \frac{1}{n} \right) | \, m < \frac{4}{\varepsilon}, n < \frac{4}{\varepsilon} \right\}. \tag{3}$$

Clearly Q is a simple graph with $\mu(Q) = \frac{\varepsilon}{2} < \varepsilon$. So $\mu(A) = 0$.

Question 3. *Let $f(x)$ be Riemann integrable on $[a,b]$. Let $A := \{(x, f(x)) | \, x \in [a,b]\}$. Prove that $\mu(A) = 0$. Is the converse – $\mu(A) = 0 \Longrightarrow f$ Riemann integrable – true? Justify.*

Solution. Since f is Riemann integrable then for any $\varepsilon > 0$ there is a partition $P = \{x_0 = a < x_1 < \cdots < x_m = b\}$ such that

$$\sum_{j=0}^{m-1} \left(\sup_{x \in [x_j, x_{j+1}]} f(x) \right) (x_{j+1} - x_j) - \sum_{j=1}^{m-1} \left(\inf_{x \in [x_j, x_{j+1}]} f(x) \right) (x_{j+1} - x_j) < \varepsilon. \tag{4}$$

But if we denote for each $j \in \{0, 1, 2, ..., m-1\}$,

$$J_j := [x_j, x_{j+1}] \times \left[\inf_{x \in [x_j, x_{j+1}]} f(x), \sup_{x \in [x_j, x_{j+1}]} f(x) \right]. \tag{5}$$

We have

$$A \subseteq \cup_{j=0}^{m-1} J_j \tag{6}$$

which leads to

$$\bar{A} \subseteq \cup_{j=0}^{m-1} J_j \tag{7}$$

since $\cup_{j=0}^{m-1} J_j$ is closed. Now (4) gives

$$\sum_{j=1}^{m-1} \mu(J_j) < \varepsilon. \tag{8}$$

Thus $\mu(A) = 0$.

The converse is not true. For example let $f(x)$ be the Dirichlet function. It is not integrable but its graph has closure $[0,1] \times (\{0\} \cup \{1\})$ which clearly has Jordan measure 0. However see the following problem.

Question 4. *Let $f(x) \geqslant 0$ be Riemann integrable on $[a,b]$ and let $A := \{(x,y) | x \in [a,b], y \leqslant f(x)\}$. Prove that A is Jordan measurable and*

$$\mu(A) = \int_0^1 f(x)\,\mathrm{d}x. \tag{9}$$

Is the converse – A is Jordan measurable $\Longrightarrow$ f is Riemann integrable – true? Justify.

Solution. From Question 4 we see that $\mu(\partial A) = 0$ so A is Jordan measurable. Define

$$g(x) = f(x) + 1, \qquad B := \{(x,y) | x \in [a,b], y \leqslant g(x)\}. \tag{10}$$

Then it suffices to prove

$$\mu(B) = \int_0^1 g(x)\,\mathrm{d}x. \tag{11}$$

Take any partition of $[a,b]$, $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$. Fix any $0 < \varepsilon < 1/2$. For every $i = 0, \ldots, n-1$, Define

$$I_i := [x_i + \varepsilon\,(x_{i+1} - x_i), x_{i+1} - \varepsilon\,(x_{i+1} - x_i)] \times \left[\varepsilon, \inf_{[x_i, x_{i+1}]} g - \varepsilon\right]. \tag{12}$$

We see that $I_i \subseteq B^o$ and

$$\mu(I_i) = (1 - 2\,\varepsilon)\,(x_{i+1} - x_i)\left(\inf_{[x_i, x_{i+1}]} g - 2\,\varepsilon\right). \tag{13}$$

This means

$$\begin{aligned}
\mu(B) &\geqslant \mu(\cup I_i) \\
&\geqslant \sum \mu(I_i) \\
&= (1 - 2\,\varepsilon) \sum_i (x_{i+1} - x_i)\left(\inf_{[x_i, x_{i+1}]} g - 2\,\varepsilon\right) \\
&= (1 - 2\,\varepsilon)\left[\sum_i \left(\inf_{[x_i, x_{i+1}]} g\right)(x_{i+1} - x_i) - 2\,\varepsilon\,(b - a)\right].
\end{aligned} \tag{14}$$

By the arbitrariness of ε we have

$$\mu(B) \geqslant \sum_i \left(\inf_{[x_i, x_{i+1}]} g\right)(x_{i+1} - x_i). \tag{15}$$

Taking supreme over all partitions we have

$$\mu(B) \geqslant \int_a^b g(x)\mathrm{d}x. \tag{16}$$

Similarly we have

$$\mu(B) \leqslant \sum_i \left(\sup_{[x_i, x_{i+1}]} g\right)(x_{i+1} - x_i), \tag{17}$$

taking infimum we have

$$\mu(B) \leqslant \int_a^b g(x)\,\mathrm{d}x. \tag{18}$$

Thus ends the proof.

The converse is true this time. Assume A is Jordan measurable. Then for every $\varepsilon > 0$ there are simple graphs B, C such that $B \subseteq A^o$, $\bar{A} \subseteq C$, with $\mu(B) \geqslant \mu(A) - \varepsilon$, $\mu(C) \leqslant \mu(A) + \varepsilon$.

Now write $B = \cup_{i=1}^n I_i$ with $I_i^o \cap I_j^o = \varnothing$. Denote $I_i = [a_i, b_i] \times [c_i, d_i]$. Then we have

$$I_i \subseteq A \Longrightarrow [a_i, b_i] \subseteq [a, b], [c_i, d_i] \subseteq \left[0, \inf_{x \in [a_i, b_i]} f(x)\right]. \tag{19}$$

Define

$$h(x) := \max_{i=1,\ldots,n, x \in [a_i, b_i]} d_i. \tag{20}$$

Note that the maximum is taken over all i's such that $x \in [a_i, b_i]$. Also note that h is a piecewise constant function.

Now define

$$B' := \{(x, y) | \, x \in [a, b], 0 \leqslant y \leqslant h(x)\}. \tag{21}$$

Then we have $h(x) \leqslant f(x)$ and

$$B \subseteq B' \subseteq A, \qquad \mu(B') = \int_a^b h(x)\,\mathrm{d}x \leqslant L(f, [a, b]) \tag{22}$$

where $L(f, [a, b])$ is the lower integral of f over $[a, b]$. This gives

$$\mu(A) - \varepsilon \leqslant \mu(B) \leqslant \mu(B') = \int_a^b h(x)\,\mathrm{d}x \leqslant L(f, [a, b]). \tag{23}$$

Similarly we can prove

$$\mu(A) + \varepsilon \geqslant \mu(C) \geqslant U(f, [a, b]). \tag{24}$$

Thus

$$U(f, [a, b]) - L(f, [a, b]) \leqslant 2\,\varepsilon \tag{25}$$

for any $\varepsilon > 0$. Consequently $U(f, [a, b]) = L(f, [a, b]) = \mu(A)$ and the conclusion follows.

Question 5. *Find a bounded open set that is not Jordan measurable. Justify your answer.*

Solution. List all rational numbers in $[0, 1]$ and $r_1, r_2, \ldots$. Define

$$A := \left[\cup_{i=1}^{\infty} \left(r_i - \frac{1}{2^{i+2}}, r_i + \frac{1}{2^{i+2}}\right)\right] \cap [0, 1]. \tag{26}$$

Clearly A is open. Assume A is Jordan measurable. Then since $\bar{A} = [0, 1]$ we have $\mu(A) = 1$.

On the other hand, since A is Jordan measurable, there is a simple graph $B \subseteq A^o = A$, with $\mu(B) \geqslant \frac{3}{4}$. Note that B is compact, so there are finitely many $i_1, \ldots, i_n$ such that

$$B \subseteq \cup_{k=1}^n \left(r_{i_k} - \frac{1}{2^{i_k+2}}, r_{i_k} + \frac{1}{2^{i_k+2}}\right). \tag{27}$$

This means

$$\mu(B) \leqslant \sum_{k=1}^n \frac{1}{2^{i_k+1}} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}. \tag{28}$$

Contradiction.

Question 6. *Prove* ***by definition*** *that* $f(x,y)=\sin(x\,y)$ *is Riemann integrable on* $I:=[0,1]\times[0,1]$.

Solution. We try to prove $U(f,I)=L(f,I)$.

For any $n\in\mathbb{N}$, set $I_{ij}:=\left[\frac{i}{n},\frac{i+1}{n}\right]\times\left[\frac{j}{n},\frac{j+1}{n}\right]$ for $i,j\in\{0,1,2,\ldots,n-1\}$. Let

$$f_{ij}:=\min_{(x,y)\in I_{ij}} f(x,y), \qquad F_{ij}:=\max_{(x,y)\in I_{ij}} F(x,y). \tag{29}$$

Since $f(x,y)=\sin(x\,y)$ is continuous over I_{ij}, there are $(x_1,y_1),(x_2,y_2)$ such that

$$\sin(x_1\,y_1)=f_{ij}, \qquad \sin(x_2\,y_2)=F_{ij}. \tag{30}$$

Now (single variable) MVT gives

$$|F_{ij}-f_{ij}|=|\cos(\xi)|\,|x_2\,y_2-x_1\,y_1|\leqslant|x_2-x_1|\,|y_2|+|y_2-y_1|\,|x_1|\leqslant\frac{2}{n}. \tag{31}$$

Next define simple functions:

$$h_n(x,y):=\min_{i,j} f_{ij}\,1_{I_{ij}}(x,y); \qquad g_n(x,y):=\max_{i,j} F_{ij}\,1_{I_{ij}}(x,y). \tag{32}$$

We have $h_n(x,y)\leqslant f(x,y)\leqslant g_n(x,y)$ on I and furthermore

$$\int_I [g_n(x,y)-h_n(x,y)]\,\mathrm{d}(x,y)=\sum_{i,j}(F_{ij}-f_{ij})\frac{1}{n^2}\leqslant\frac{2}{n}. \tag{33}$$

This gives

$$U(f,I)-L(f,I)\leqslant\frac{2}{n}. \tag{34}$$

Since $U(f,I)-L(f,I)\geqslant 0$ by definition, the arbitrariness of n now gives $U(f,I)-L(f,I)=0$ and integrability follows.