Math 217 Fall 2013 Homework 7 Solutions

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let* $f(x, y) = x^3 + y^3 + x\, y^2$*. Calculate its Taylor expansion to degree* 2 *with remainder (that is* $n = 2$*, the remainder involves 3rd order derivatives) at* $(1, 0)$*.*

Question 2. *Let* $f(x, y) = \frac{x^2}{y}$*. Calculate its Taylor polynomial of degree* 3 *(that is* P_3*) at* $(1, 1)$*.*

Question 3. *Let* $f(x, y, z) = \frac{\cos x \cos y}{\cos z}$*. Calculate its Hessian matrix at* $(0, 0, 0)$*.*

Question 4. *Let* $f: \mathbb{R}^N \mapsto \mathbb{R}$ *belong to* C^2*, that is all of its second order partial derivatives exist and are continuous. Let* $\boldsymbol{x}_0 \in \mathbb{R}^N$*. Assume*

$$\forall \boldsymbol{v} \in \mathbb{R}^N, \boldsymbol{v} \neq \boldsymbol{0} \quad \boldsymbol{v}^T H(\boldsymbol{x}_0)\, \boldsymbol{v} > 0 \tag{1}$$

where $H(\boldsymbol{x}_0)$ *is the Hessian matrix of* f *at* $\boldsymbol{x}_0$*. Prove that there is* $r > 0$ *such that for all* $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$*, there holds*

$$\forall \boldsymbol{v} \in \mathbb{R}^N, \boldsymbol{v} \neq \boldsymbol{0} \quad \boldsymbol{v}^T H(\boldsymbol{x})\, \boldsymbol{v} > 0. \tag{2}$$

Question 5. *Prove*

$$a, b \geqslant 0, n \geqslant 1 \Longrightarrow \left(\frac{a+b}{2}\right)^n \leqslant \frac{a^n + b^n}{2} \tag{3}$$

through solving $f(x, y) = x^n + y^n$ *subject to the constraint* $x + y = l > 0$*.*

Question 6. *Let* $f: \mathbb{R}^N \mapsto \mathbb{R}$ *belong to* C^2*. Let* $\boldsymbol{x}_0 \in \mathbb{R}^N$ *be a local* ***maximizer*** *for* f*. Prove*

a) $(\operatorname{grad} f)(\boldsymbol{x}_0) = \boldsymbol{0}$*;*

b) $\forall \boldsymbol{v} \in \mathbb{R}^N$*,* $\boldsymbol{v}^T H(\boldsymbol{x}_0)\, \boldsymbol{v} \leqslant 0$ *where* $H(\boldsymbol{x}_0)$ *is the Hessian matrix of* f *at* $\boldsymbol{x}_0$*.*