Math 217 Fall 2013 Homework 6 Solutions

Due Thursday Oct. 31, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $A \subseteq \mathbb{R}^N$ be convex. Let $r > 0$ and $B := \{\boldsymbol{x} \in \mathbb{R}^N | \operatorname{dist}(\boldsymbol{x}, A) < r\}$, where $\operatorname{dist}(\boldsymbol{x}, A) := \inf_{\boldsymbol{y} \in A} \|\boldsymbol{x} - \boldsymbol{y}\|$. Prove that B is convex.*

Solution. Take any $\boldsymbol{x}, \boldsymbol{y} \in B$ and $t \in [0, 1]$. We need to show $\operatorname{dist}(t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}, A) < r$, that is, we need to find $\boldsymbol{z} \in A$ such that $\operatorname{dist}(t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}, \boldsymbol{z}) < r$. Sincer $\boldsymbol{x}, \boldsymbol{y} \in B$, there are $\boldsymbol{u}, \boldsymbol{v} \in A$ such that $\|\boldsymbol{x} - \boldsymbol{u}\| < r$, $\|\boldsymbol{y} - \boldsymbol{v}\| < r$. Then we have

$$\begin{aligned} \|(t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}) - (t\,\boldsymbol{u} + (1-t)\,\boldsymbol{v})\| &= \|t\,(\boldsymbol{x} - \boldsymbol{u}) + (1-t)\,(\boldsymbol{y} - \boldsymbol{v})\| \\ &\leqslant \|t\,(\boldsymbol{x} - \boldsymbol{u})\| + \|(1-t)\,(\boldsymbol{y} - \boldsymbol{v})\| \\ &= t\,\|\boldsymbol{x} - \boldsymbol{u}\| + (1-t)\,\|\boldsymbol{y} - \boldsymbol{v}\| \\ &< r. \end{aligned} \tag{1}$$

Since A is convex, $t\,\boldsymbol{u} + (1-t)\,\boldsymbol{v} \in A$. Thus ends the proof.

Question 2. *Consider $\boldsymbol{f}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined through*

$$\boldsymbol{f}(x, y) := \begin{pmatrix} x^3 - 3\,x\,y^2 \\ 3\,x^2\,y - y^3 \end{pmatrix}. \tag{2}$$

Prove:

a) *For every $(x_0, y_0) \neq (0, 0)$ there is open set $U \ni (x_0, y_0)$ such that $\boldsymbol{f}$ is one-to-one on U;*

b) *Let U be open and $(0, 0) \in U$. Then $\boldsymbol{f}$ is not one-to-one on U.*

Solution.

a) We calculate the Jacobian matrix

$$\frac{\partial \boldsymbol{f}}{\partial(x, y)} = \begin{pmatrix} 3\,x^2 - 3\,y^2 & -6\,x\,y \\ 6\,x\,y & 3\,x^2 - 3\,y^2 \end{pmatrix}. \tag{3}$$

Therefore

$$\det\left(\frac{\partial \boldsymbol{f}}{\partial(x, y)}\right) = 9\,(x^2 - y^2)^2 + 36\,x^2\,y^2. \tag{4}$$

Now if the determinant is 0, then necessarily $x^2 = y^2$ and $x^2\,y^2 = 0$ which means $x^2 = y^2 = 0$. Therefore the determinant is not 0 at every $(x_0, y_0) \neq (0, 0)$.

Since the partial derivatives of $\boldsymbol{f}$ are polynomials they are continuous. Application of Implicit function theorem gives the desired result.

b) For any $r > 0$, let

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{r}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{r}{2}\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{5}$$

Then both points are inside $B(\mathbf{0}, r)$ but

$$\boldsymbol{f}(x_1, y_1) = \frac{r^3}{8}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{f}(x_2, y_2) = \frac{r^3}{8}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6}$$

Therefore $\boldsymbol{f}$ is not one-to-one.

Question 3. *Let $f\colon (a, b) \mapsto \mathbb{R}$ such that $f^{(n+1)}(x)$ is continuous. For any $x_0, x \in (a, b)$, write*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0). \tag{7}$$

a) *Prove that $\frac{\partial R_n}{\partial x_0}$ exists;*

b) *Calculate $\frac{\partial R_n}{\partial x_0}$;*

c) *Prove that*

$$R_n(x, x_0) = \frac{1}{n!}\int_{x_0}^{x} (x - y)^n f^{(n+1)}(y)\, \mathrm{d}y. \tag{8}$$

Solution.

a) Since

$$R_n(x, x_0) = f(x) - \left[f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right] \tag{9}$$

and each term on the RHS is partially differentiable with respect to x_0, $\frac{\partial R_n}{\partial x_0}$ exists.

b) Taking $\frac{\partial}{\partial x_0}$ of both sides of (9), we have

$$\begin{aligned} \frac{\partial R_n(x, x_0)}{\partial x_0} &= 0 - f'(x_0) - [f''(x_0)(x - x_0) - f'(x_0)] \\ &\quad - \left[\frac{f'''(x_0)}{2!}(x - x_0)^2 - \frac{f''(x_0)}{1!}(x - x_0) \right] \\ &\quad - \cdots - \left[\frac{f^{(n+1)}(x_0)}{n!}(x - x_0)^n - \frac{f^{(n)}(x_0)}{(n-1)!}(x - x_0)^{n-1} \right] \\ &= -\frac{f^{(n+1)}(x_0)}{n!}(x - x_0)^n. \end{aligned} \tag{10}$$

c) Now we have proved:

$$\frac{\partial R_n(x, y)}{\partial y} = -\frac{f^{(n+1)}(y)}{n!}(x - y)^n. \tag{11}$$

Now for any fixed $x \in (a, b)$, since by assumption $f^{(n+1)}(y)$ is continuous, $\frac{\partial R_n(x, y)}{\partial y}$ is continuous as a function of y. Application of fundamental theorem of calculus then gives

$$R(x, x) - R(x, x_0) = \int_{x_0}^{x} \frac{\partial R_n(x, y)}{\partial y}\, \mathrm{d}y = -\frac{1}{n!}\int_{x_0}^{x} (x - y)^n f^{(n+1)}(y)\, \mathrm{d}y. \tag{12}$$

The proof ends through noticing

$$R(x, x) = 0. \tag{13}$$

Question 4. *Calculate all third order partial derivartives for $f(x, y) = x^4 + y^4 + 4x^2y^2$.*

Solution. We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4\,x^3+8\,x\,y^2;\\ \frac{\partial f}{\partial y} &= 4\,y^3+8\,x^2\,y;\\ \frac{\partial^2 f}{\partial x^2} &= 12\,x^2+8\,y^2;\\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x} &= 16\,x\,y;\\ \frac{\partial^2 f}{\partial y^2} &= 12\,y^2+8\,x^2;\\ \frac{\partial^3 f}{\partial x^3} &= 24\,x;\\ \frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial^3 f}{\partial x\partial y\partial x}=\frac{\partial^3 f}{\partial y\partial x^2} &= 16\,y;\\ \frac{\partial^3 f}{\partial x\partial y^2}=\frac{\partial^3 f}{\partial y\partial x\partial y}=\frac{\partial^3 f}{\partial y^2\partial x} &= 16\,x;\\ \frac{\partial^3 f}{\partial y^3} &= 24\,y.\end{aligned}$$

Question 5. *Let* $f(x,y,z):=x\,y\,z\,e^{x+y+z}$. *Find* $\frac{\partial^{p+q+r} f}{\partial x^p\partial y^q\partial z^r}$. *Here* $p,q,r\in\mathbb{N}\cup\{0\}$.

Solution. We notice that

$$f(x,y,z)=(x\,e^x)\,(y\,e^y)\,(z\,e^z). \tag{14}$$

Thus

$$\frac{\partial^{p+q+r} f}{\partial x^p\partial y^q\partial z^r}=\left[\frac{\mathrm{d}^p}{\mathrm{d}x^p}(x\,e^x)\right]\left[\frac{\mathrm{d}^q}{\mathrm{d}y^q}(y\,e^y)\right]\left[\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z\,e^z)\right]. \tag{15}$$

Now we prove

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}(x\,e^x)=(x+p)\,e^x \tag{16}$$

using induction.

The base case $p=0$ is obvious.

Assume $\frac{\mathrm{d}^p}{\mathrm{d}x^p}(x\,e^x)=(x+p)\,e^x$. We have

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}x^{p+1}}(x\,e^x)=\frac{\mathrm{d}}{\mathrm{d}x}[(x+p)\,e^x]=(x+p+1)\,e^x. \tag{17}$$

Therefore

$$\frac{\partial^{p+q+r} f}{\partial x^p\partial y^q\partial z^r}=(x+p)\,(y+q)\,(z+r)\,e^{x+y+z}. \tag{18}$$

Question 6. *Let* $f:\mathbb{R}^2\mapsto\mathbb{R}$. *Assume* $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ *are differentiable at* (x_0,y_0). *Then*

$$\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)=\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0). \tag{19}$$

Solution. Let $r>0$ be such that $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ exist in $B((x_0,y_0),r)$. Consider any $(x,y)\in B((x_0,y_0),r)$. Let

$$A:=f(x,y)-f(x_0,y)-[f(x,y_0)-f(x_0,y_0)]. \tag{20}$$

Setting $\varphi(y) := f(x,y) - f(x_0,y)$ and apply MVT, we have

$$A = \left[\frac{\partial f}{\partial y}(x,\eta) - \frac{\partial f}{\partial y}(x_0,\eta)\right](y - y_0). \tag{21}$$

Since $\frac{\partial f}{\partial y}$ is differentiable at (x_0, y_0),

$$\frac{\partial f}{\partial y}(x,\eta) - \frac{\partial f}{\partial y}(x_0,y_0) = \left(\text{grad}\,\frac{\partial f}{\partial y}\right)(x_0,y_0)\cdot\begin{pmatrix} x-x_0 \\ \eta - y_0\end{pmatrix} + R_1, \tag{22}$$

$$\frac{\partial f}{\partial y}(x_0,\eta) - \frac{\partial f}{\partial y}(x_0,y_0) = \left(\text{grad}\,\frac{\partial f}{\partial y}\right)(x_0,y_0)\cdot\begin{pmatrix} x_0-x_0 \\ \eta - y_0\end{pmatrix} + R_2, \tag{23}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|R_1| + |R_2|}{\|(x-x_0, y-y_0)\|} = 0. \tag{24}$$

Taking difference we have

$$\frac{\partial f}{\partial y}(x,\eta) - \frac{\partial f}{\partial y}(x_0,\eta) = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\,(x-x_0) + R \tag{25}$$

and therefore

$$A = \frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)\,(x-x_0)\,(y-y_0) + R\,(y-y_0); \tag{26}$$

where R a function of x, y, x_0, y_0, f and satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R}{\|(x-x_0, y-y_0)\|} = 0. \tag{27}$$

Similarly writing

$$A = f(x,y) - f(x,y_0) - [f(x_0,y) - f(x_0,y_0)] \tag{28}$$

we have

$$A = \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\,(x-x_0)\,(y-y_0) + R'(x-x_0) \tag{29}$$

with R' a function of x, y, x_0, y_0, f and satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R'}{\|(x-x_0, y-y_0)\|} = 0. \tag{30}$$

Summarizing, we have

$$\left[\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) - \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right] = \frac{R'\,(x-x_0) + R\,(y-y_0)}{(x-x_0)\,(y-y_0)}. \tag{31}$$

Note that this holds for all $(x,y) \in B((x_0,y_0), r)$.

Now taking $x = x_0 + t$, $y = y_0 + t$, and letting $t \longrightarrow 0$, we have $t = \frac{1}{\sqrt{2}}\,\|(x,y) - (x_0,y_0)\|^2$ and then

$$\begin{aligned}\left[\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) - \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right] &= \lim_{t\longrightarrow 0}\left[\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) - \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right] \\ &= \lim_{t\to 0}\frac{R' + R}{t} \\ &= 0. \end{aligned} \tag{32}$$

Thus ends the proof.