MATH 217 FALL 2013 HOMEWORK 5

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Consider* $\boldsymbol{f}: \mathbb{R} \mapsto \mathbb{R}^3$ *defined through*

$$\boldsymbol{f}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}. \tag{1}$$

Find $t_1 < t_2$ *such that there is no* $\xi \in (t_1, t_2)$ *satisfying*

$$\boldsymbol{f}(t_2) - \boldsymbol{f}(t_1) = \boldsymbol{f}'(\xi)\,(t_2 - t_1). \tag{2}$$

Explain why this is not contradicting Mean Value Theorem.

Question 2. *Find* $f(x, y)$ *such that* f *is differentiable (meaning differentiable everywhere) but* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *are not continuous.*

Question 3. *Let* $f(x, y) = \sin(2x)\sin y \sin(2x + y)$ *and* $A := \{(x, y) | x \geqslant 0, y \geqslant 0, 2x + y \leqslant \pi\}$. *Find* $\max_{(x,y)\in A} f(x, y)$.

Question 4. *Let* $z = Z(x, y)$ *be determined through the equation*

$$x\,y + y\,z + z\,x = 1. \tag{3}$$

Find $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$ *without solving* Z *explicitly.*

Question 5. *Let* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N \in \mathbb{R}^N$ *be such that* $\|\boldsymbol{v}_i\| = 1$ *for all* i, $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ *for all* $i \neq j$. *Let* $f: \mathbb{R}^N \mapsto \mathbb{R}$ *be differentiable. Prove*

$$\left(\frac{\partial f}{\partial \boldsymbol{v}_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial \boldsymbol{v}_N}\right)^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_N}\right)^2. \tag{4}$$

Question 6. *Let* $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$ *have continuous partial derivatives. Let* $\alpha > 0$. *Assume that* $\boldsymbol{f}$ *satisfies*

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \geqslant \alpha\,\|\boldsymbol{x} - \boldsymbol{y}\| \tag{5}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. *Prove*

a) $\det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \neq 0$ *for all* $\boldsymbol{x}$;

b) For any fixed $\boldsymbol{y}_0 \in \mathbb{R}^N$, $F(\boldsymbol{x}) := \|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})\|$ *reaches minimum but not maximum;*

c) $\boldsymbol{f}(\mathbb{R}^N) = \mathbb{R}^N$.