Math 217 Fall 2013 Homework 4 Solutions

Due Thursday Oct. 10, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $f:[0,1)\times[0,1)$ be defined as*

$$f(x,y)=\frac{1}{1-x\,y}. \tag{1}$$

Prove that f is continuous (not necessarily by definition) but not uniformly continuous.

Solution.

- f is continuous. This follows directly from the fact that f is the ratio of two continuous functions $1, 1-x\,y$ and $1-x\,y\neq 0$ for all $(x,y)\in[0,1)\times[0,1)$.
- f is not uniformly continuous. For any $1>\delta>0$, let $x_1=1-\delta/2, x_2=1-\delta$. Then

  $$\|(x_1,x_1)-(x_2,x_2)\|=\frac{\sqrt{2}}{2}\,\delta<\delta, \tag{2}$$

  but

  $$|f(x_1,x_1)-f(x_2,x_2)|=\frac{1}{\delta-\delta^2/4}-\frac{1}{2\,\delta-\delta^2}>\frac{1}{\delta/2-\delta^2/4}-\frac{1}{2\,\delta-\delta^2}>\frac{1}{\delta-\delta^2/2}>2. \tag{3}$$

Question 2. *Prove by definition (without using Heine-Borel):*

a) $E=\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\}\subseteq\mathbb{R}^N$ is compact;

b) $E=\{(x,y)|\,x\in\mathbb{N}, y\in\mathbb{N}\}$ is not compact;

Proof.

a) Let W be any open covering of E. Then for each $i\in\{1,2,\ldots,n\}$, since $\boldsymbol{x}_i\in\cup_{O\in W}O$, there is $O_i\in W$ such that $\boldsymbol{x}_i\in O_i$. Now we have the desired finite covering

$$E\subseteq\cup_{i=1}^n O_i. \tag{4}$$

b) Consider the open covering

$$E\subseteq\cup_{i,j=1}^\infty O_{ij} \tag{5}$$

where

$$O_{ij}:=B((i,j),1). \tag{6}$$

Then we see that each $(i,j)\in E$ satisfies $(i,j)\in O_{ij}$ but

$$\forall(k,l)\neq(i,j)\qquad (i,j)\notin O_{kl}. \tag{7}$$

Therefore any finite covering can only cover finitely many points in E and cannot cover E. □

Question 3. *Let $\boldsymbol{f}:\mathbb{R}^N\mapsto\mathbb{R}^M$ be continuous. Let $A\subseteq\mathbb{R}^N$.*

a) Prove that $\boldsymbol{f}(\overline{A})\subseteq\overline{\boldsymbol{f}(A)}$;

b) *Give an example where* $\boldsymbol{f}(\overline{A}) \subset \overline{\boldsymbol{f}(A)}$ *(that is* $\boldsymbol{f}(\overline{A}) \subseteq \overline{\boldsymbol{f}(A)}$ *but* $\boldsymbol{f}(\overline{A}) \neq \overline{\boldsymbol{f}(A)}$*).*

c) *What is the weakest additional assumption on* E *you can find that guarantees* $\boldsymbol{f}(\overline{A}) = \overline{\boldsymbol{f}(A)}$ *for all continuous* $\boldsymbol{f}$*? Justify your answer.*

Solution.

a) Take any $\boldsymbol{y}_0 \in \boldsymbol{f}(\overline{A})$. Then there is $\boldsymbol{x} \in \overline{A}$ such that $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x})$. Two cases:

1. $\boldsymbol{x}_0 \in A$. Then $\boldsymbol{y}_0 \in \boldsymbol{f}(A) \subseteq \overline{\boldsymbol{f}(A)}$;
2. $\boldsymbol{x}_0 \notin A$. Then we claim that for every $r > 0$, $B(\boldsymbol{x}_0, r) \cap A \neq \varnothing$. To see this, assume otherwise. Then there is $r_0 > 0$ such that $B(\boldsymbol{x}_0, r_0) \cap A = \varnothing \Longrightarrow A \subseteq B(\boldsymbol{x}_0, r_0)^c$. Now we have $A \subseteq \overline{A} \cap B(\boldsymbol{x}_0, r_0)^c \subset \overline{A}$. Note that the middle set is closed. This contradicts the definition of closure as the intersection of all closed sets containing A.

   Since $\boldsymbol{f}$ is continuous, for every $\varepsilon > 0$, there is $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \Longrightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \varepsilon$. We know that $B(\boldsymbol{x}_0, \delta) \cap A \neq \varnothing$ therefore $\boldsymbol{f}(A) \cap B(\boldsymbol{y}_0, \varepsilon) \neq \varnothing$. This is true for all $\varepsilon > 0$, so $\boldsymbol{y}_0 \in \overline{\boldsymbol{f}(A)}$.

b) Consider

$$f(\boldsymbol{x}) = e^{-\|\boldsymbol{x}\|}. \tag{8}$$

and $\overline{A} = A = \mathbb{R}^N$ and $f(A) = (0, 1]$.

c) The weakest addition condition is "$\overline{A}$ is compact".

- If $\overline{A}$ is further compact, then $\boldsymbol{f}(\overline{A}) = \overline{\boldsymbol{f}(A)}$.

  Since $\boldsymbol{f}(A) \subseteq \boldsymbol{f}(\overline{A})$, all we need to show is $\boldsymbol{f}(\overline{A})$ is closed. By Heine-Borel it suffices to show $\boldsymbol{f}(\overline{A})$ is compact. Let W be an open covering of $\boldsymbol{f}(\overline{A})$. Then

  $$W' := \{\boldsymbol{f}^{-1}(O) | O \in W\} \tag{9}$$

  is an open covering of $\overline{A}$. By compactness there is a subcovering

  $$\bar{A} \subseteq \cup_{k=1}^{n} \boldsymbol{f}^{-1}(O_k). \tag{10}$$

  Now this gives

  $$\boldsymbol{f}(\bar{A}) \subseteq \boldsymbol{f}(\cup_{k=1}^{n} \boldsymbol{f}^{-1}(O_k)) \subseteq \cup_{k=1}^{n} \boldsymbol{f}(\boldsymbol{f}^{-1}(O_k)) \subseteq \cup_{k=1}^{n} O_k. \tag{11}$$

  This is a finite covering of $\boldsymbol{f}(\overline{A})$.

- Now we show that if $\boldsymbol{f}(\overline{A}) = \overline{\boldsymbol{f}(A)}$ for all continuous $\boldsymbol{f}$, then $\overline{A}$ must be compact.

  Assume otherwise. By Heine-Borel, $\overline{A}$ is not bounded. We claim that A is not bounded either. Assume otherwise, then there is $R > 0$ such that $A \subseteq B(\mathbf{0}, R)$. Since $\overline{A}$ is not bounded, there is $\boldsymbol{x} \in \overline{A}$ such that $\|\boldsymbol{x}\| > R + 1$. Note that $\boldsymbol{x} \notin A$. Now consider the set

  $$B := \overline{A} \cap B(\boldsymbol{x}, 1)^c. \tag{12}$$

  Then clearly

  1. B is closed;
  2. $A \subseteq B \subset \overline{A}$.

  This contradicts the fact that $\overline{A}$ is the intersection of all closed sets containing A.

  Now define $f(\boldsymbol{x}) := \exp(-\|\boldsymbol{x}\|)$ and obviously $0 \in \overline{f(A)}$ but $0 \notin f(\overline{A})$.

Remark. Sets $A \subseteq \mathbb{R}^N$ satisfying $\overline{A}$ is compact are called "precompact".

Question 4. *Let $f(x,y,z)=x^2+y^2+z^2$. Prove by definition that f is differentiable at $(1,1,1)$ and find its differential there.*

Proof. Guess $D f(1,1,1)(u,v,w)=2\,u+2\,v+2\,w$.

Now set $x=1+u$, $y=1+v$, $z=1+w$ and check

$$\frac{|f(x,y,z)-f(1,1,1)-(2\,u+2\,v+2\,w)|}{(u^2+v^2+w^2)^{1/2}}=\frac{u^2+v^2+w^2}{(u^2+v^2+w^2)^{1/2}}=(u^2+v^2+w^2)^{1/2}. \tag{13}$$

Thus

$$\lim_{(u,v,w)\longrightarrow(0,0,0)}\frac{|f(x,y,z)-f(1,1,1)-(2\,u+2\,v+2\,w)|}{(u^2+v^2+w^2)^{1/2}}=0 \tag{14}$$

and the proof ends. □

Question 5. *Let $f(x,y,z)=y^2\,z+\sin(5\,x\,y)$. Calculate its three partial derivatives.*

Solution. We have

$$\frac{\partial f}{\partial x}(x,y,z)=5\,y\cos(5\,x\,y),\ \frac{\partial f}{\partial y}(x,y,z)=2\,y\,z+5\,x\cos(5\,x\,y),\ \frac{\partial f}{\partial z}(x,y,z)=y^2. \tag{15}$$

Question 6. *Let $f(x,\,y)=|x+y|$. Find all directions $\boldsymbol{v}\in\mathbb{R}^3$ such that $\frac{\partial f}{\partial \boldsymbol{v}}$ exists. Justify your answer. Note that the answer may be different at different points (x,y).*

Solution. There are two cases. $x+y=0$ and $x+y\neq 0$. We denote $\boldsymbol{v}=\begin{pmatrix} u \\ v \end{pmatrix}$.

1. $x+y=0$. In this case $\frac{\partial f}{\partial \boldsymbol{v}}(x,y)$ exists if and only if $u+v=0$.
   - If. In this case we have
   $$f((x,y)+h\,(u,v))=f(x,y) \tag{16}$$
   so obviously
   $$\frac{\partial f}{\partial \boldsymbol{v}}(x,y)=0. \tag{17}$$
   - Only if. We show that if $u+v\neq 0$, then $\frac{\partial f}{\partial \boldsymbol{v}}(x,y)$ does not exist.
     Wlog assume $u+v>0$. Then since $x+y=0$, we have
   $$f((x,y)+h\,(u,v))=\begin{cases} h\,(u+v) & h>0 \\ -h\,(u+v) & h<0 \end{cases}=|h|\,(u+v) \tag{18}$$
   The limit
   $$\lim_{h\longrightarrow 0}\frac{|h|\,(u+v)}{h} \tag{19}$$
   doesn't exist.
2. $x+y\neq 0$. In this case $\frac{\partial f}{\partial \boldsymbol{v}}(x,y)$ for all directions $\boldsymbol{v}$.
   Wlog assume $x+y=:\delta>0$. Now for all $|h|<\frac{\delta}{\sqrt{2}\,\|\boldsymbol{v}\|}$, we have by Cauchy-Schwarz,
   $$|u+v|=|1\cdot u+1\cdot v|\leqslant\sqrt{2}\,(u^2+v^2)^{1/2}. \tag{20}$$
   This gives
   $$|h\,u+h\,v|\leqslant\sqrt{2}\,|h|\,\|\boldsymbol{v}\|<\delta \tag{21}$$
   and consequently
   $$(x+h u)+(y+h\,v)>0\Longrightarrow f(x+h\,u,y+h\,v)=(x+h u)+(y+h\,v). \tag{22}$$
   Thus
   $$\lim_{h\longrightarrow 0}\frac{f(x+h\,u,y+h\,v)-f(x,y)}{h}=\lim_{h\longrightarrow 0}(u+v)=u+v. \tag{23}$$