Math 217 Fall 2013 Homework 4 Solutions

Due Thursday Oct. 10, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $f\colon [0,1) \times [0,1)$ be defined as*

$$f(x,y) = \frac{1}{1 - x\,y}. \tag{1}$$

Prove that f is continuous (not necessarily by definition) but not uniformly continuous.

Question 2. *Prove by definition (without using Heine-Borel):*

a) $E = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^N$ *is compact;*

b) $E = \{(x,y) |\, x \in \mathbb{N}, y \in \mathbb{N}\}$ *is not compact;*

Question 3. *Let $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$ be continuous. Let $A \subseteq \mathbb{R}^N$.*

a) Prove that $\boldsymbol{f}(\overline{A}) \subseteq \overline{\boldsymbol{f}(A)}$*;*

b) Give an example where $\boldsymbol{f}(\overline{A}) \subset \overline{\boldsymbol{f}(A)}$ *(that is* $\boldsymbol{f}(\overline{A}) \subseteq \overline{\boldsymbol{f}(A)}$ *but* $\boldsymbol{f}(\overline{A}) \neq \overline{\boldsymbol{f}(A)}$*).*

c) What is the weakest additional assumption on E you can find that guarantees $\boldsymbol{f}(\overline{A}) = \overline{\boldsymbol{f}(A)}$ *for all continuous $\boldsymbol{f}$? Justify your answer.*

Question 4. *Let $f(x,y,z) = x^2 + y^2 + z^2$. Prove by definition that f is differentiable at $(1,1,1)$ and find its differential there.*

Question 5. *Let $f(x,y,z) = y^2\,z + \sin(5\,x\,y)$. Calculate its three partial derivatives.*

Question 6. *Let $f(x,\,y) = |x + y|$. Find all directions $\boldsymbol{v} \in \mathbb{R}^3$ such that $\frac{\partial f}{\partial \boldsymbol{v}}$ exists. Justify your answer. Note that the answer may be different at different points (x,y).*