Math 217 Fall 2013 Homework 3 Solutions

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. (Convexity)

a) *Let $E \subset \mathbb{R}^N$ be defined by*

$$E := \{\boldsymbol{x} \in \mathbb{R}^N | \, \|\boldsymbol{x}\| < 1\} \cup \{(1, 0, \ldots, 0)\}. \tag{1}$$

Is E convex? Justify your answer.

b) *Let $S \subseteq S(\mathbf{0}, 1) := \{\boldsymbol{x} \in \mathbb{R}^N | \, \|\boldsymbol{x}\| = 1\}$ be any subset of the unit sphere. Define*

$$E := \{\boldsymbol{x} \in \mathbb{R}^N | \, \|\boldsymbol{x}\| < 1\} \cup S. \tag{2}$$

Is E convex? Justify your answer.

Solution.

a) Yes. Take any $\boldsymbol{x}, \boldsymbol{y} \in E$. Let $t \in [0, 1]$ be arbitrary.

We discuss two cases.

- Case 1: Both $\boldsymbol{x}, \boldsymbol{y} \neq (1, 0, \ldots, 0)$. Then $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| < 1$ and triangle inequality gives

$$\|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\| \leqslant \|t\,\boldsymbol{x}\| + \|(1-t)\,\boldsymbol{y}\| < t + (1-t) = 1. \tag{3}$$

Therefore $t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in E$.

- Case 2: One of $\boldsymbol{x}, \boldsymbol{y} = (1, 0, \ldots, 0)$. Wlog assume it's $\boldsymbol{x}$. Then $\|\boldsymbol{y}\| < 1$. Note that since $\boldsymbol{x}, \boldsymbol{y} \in E$. We only need to show

$$t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in E \tag{4}$$

for all $t \in (0, 1)$. This implies

$$\|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\| \leqslant \|t\,\boldsymbol{x}\| + \|(1-t)\,\boldsymbol{y}\| < t + (1-t) < 1. \tag{5}$$

Therefore $t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in E$.

b) Yes. Note that the difficulty here is that both $\|\boldsymbol{x}\|, \|\boldsymbol{y}\|$ may be 1 and the simple application of triangle inequality giving

$$\|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\| \leqslant \|t\,\boldsymbol{x}\| + \|(1-t)\,\boldsymbol{y}\| \leqslant t + (1-t) = 1. \tag{6}$$

is not enough to conclude $t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in E$.

Thus we try to prove that if $\boldsymbol{x} \neq \boldsymbol{y}$, $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$, and $t \in (0, 1)$, then $\|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\| < 1$.[1] We check:

$$\begin{aligned} \|t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}\|^2 &= [t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}] \cdot [t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}] \\ &= t^2\,\boldsymbol{x} \cdot \boldsymbol{x} + 2\,t\,(1-t)\,\boldsymbol{x} \cdot \boldsymbol{y} + (1-t)^2\,\boldsymbol{y} \cdot \boldsymbol{y} \\ &\leqslant [t^2 + (1-t)^2] + 2\,t\,(1-t)\,\boldsymbol{x} \cdot \boldsymbol{y}. \end{aligned} \tag{7}$$

1. This is a property of the norm itself. Such norms are called "strictly convex".

Now recall that

$$(x_1\, y_1 + \cdots + x_N\, y_N)^2 = (x_1^2 + \cdots + x_N^2)\,(y_1^2 + \cdots + y_N^2) - \sum_{i \neq j} (x_i\, y_j - x_j\, y_i)^2 \tag{8}$$

which means

$$|\boldsymbol{x} \cdot \boldsymbol{y}| < \|\boldsymbol{x}\|\, \|\boldsymbol{y}\| = 1 \tag{9}$$

unless

$$x_i\, y_j = x_j\, y_i \tag{10}$$

for all $i \neq j$. Taking square and sum over i, using the fact that $\sum_{i=1}^N x_i^2 = \sum_{i=1}^N y_i^2 = 1$ we reach

$$y_j^2 = x_j^2 \qquad \forall j = 1, 2, \ldots, N. \tag{11}$$

Now reviewing $x_i\, y_j = x_j\, y_i$ we see that there are only two cases, either $\boldsymbol{x} = \boldsymbol{y}$ or $\boldsymbol{x} = -\boldsymbol{y}$. The former is excluded by assumption. In the latter case, we have

$$\boldsymbol{x} \cdot \boldsymbol{y} = -\|\boldsymbol{x}\|^2 = -1 < 1. \tag{12}$$

Thus we have show that

$$\|t\, \boldsymbol{x} + (1-t)\, \boldsymbol{y}\| < 1 \tag{13}$$

which gives $t\, \boldsymbol{x} + (1-t)\, \boldsymbol{y} \in E$ when both $\boldsymbol{x}, \boldsymbol{y} \in S$.

When $\|\boldsymbol{x}\| < 1$ or $\|\boldsymbol{y}\| < 1$ the proof is the same as in a).

Question 2. (Limit) *Let $k, l, m, n \in \mathbb{N}$. Consider the following function:*

$$f(x, y) = \frac{x^k\, y^l}{x^{2m} + y^{2n}}. \tag{14}$$

Find all k, l, m, n such that the limit $\lim_{(x,y) \longrightarrow (0,0)} f(x, y)$ exist. Justify your answer. (You may find the following Young's inequality useful: $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow |x\, y| \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q}$.)

Solution. We claim that the limit is 0 when $\frac{k}{m} + \frac{l}{n} > 2$ and does not exist for all other k, l, m, n.

- $\frac{k}{m} + \frac{l}{n} > 2$. There are $r < \frac{k}{m}, s < \frac{l}{n}$ such that $r + s = 2$. Denote $\mu := \frac{k}{m} - r > 0$ and $\nu := \frac{l}{n} - s > 0$. Now apply Young's inequality:

$$|x^k\, y^l| = |x^m|^r\, |y^n|^s\, |x^{m\mu}|\, |y^{n\nu}| \leqslant |x^{m\mu}|\, |y^{n\nu}| \left(\frac{2}{r}\, x^{2m} + \frac{2}{s}\, y^{2n} \right). \tag{15}$$

This gives

$$\begin{aligned} |f(x, y)| &\leqslant \max\left(\frac{2}{r}, \frac{2}{s}\right) |x^{m\mu}|\, |y^{n\nu}| \\ &\leqslant \max\left(\frac{2}{r}, \frac{2}{s}\right) (x^2 + y^2)^{\frac{m\mu + n\nu}{2}}. \end{aligned} \tag{16}$$

Now for any $\varepsilon > 0$, take $\delta > 0$ such that

$$\max\left(\frac{2}{r}, \frac{2}{s}\right) \delta^{m\mu + n\nu} < \varepsilon. \tag{17}$$

We have whenever $\|(x, y)\| < \delta$, $|f(x, y)| < \varepsilon$.

- $\frac{k}{m} + \frac{l}{n} \leqslant 2$. We show that for every $\delta > 0$, there are $(x_1, y_1), (x_2, y_2)$ satisfying $\|(x_1, y_1)\| < \delta$, $\|(x_2, y_2)\| < \delta$, and $|f(x_1, y_1) - f(x_2, y_2)| \geqslant 1/2$.

Take any $\delta > 0$.

- Take $(x_1, y_1) = \left(\frac{\delta}{2}, 0\right)$. Then $\|(x_1, y_1)\| < \delta$ and $f(x_1, y_1) = 0$.
- Take $(x_2, y_2) = \left(t^{1/m}, t^{1/n}\right)$ where $t = \min\left(1, \frac{\delta^{m+n}}{2}\right)$. We have $\|(x_2, y_2)\| < \delta$ and

$$|f(x_2, y_2)| = \frac{1}{2} t^{2-\left(\frac{k}{m}+\frac{l}{n}\right)} \geqslant \frac{1}{2}. \tag{18}$$

Question 3. (Limit at infinity) *Let* $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$. *We define its limit at infinity as follows.* $\lim_{\boldsymbol{x} \longrightarrow \infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L} \in \mathbb{R}^M$ *if and only if*

$$\forall \varepsilon > 0\ \exists R > 0\ \forall \boldsymbol{x} \text{ satisfying } \|\boldsymbol{x}\| > R \qquad \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{L}\| < \varepsilon. \tag{19}$$

Study the limit

$$\lim_{(x,y) \longrightarrow \infty} x\, y\, e^{-x^2 y^2}. \tag{20}$$

Does it exist? If it does, what is the limit? Justify your answer.

Solution. It does not exist. For any $R > 0$, consider $(x_1, y_1) = \left(R, \frac{1}{R}\right)$ and $(x_2, y_2) = \left(R, \frac{2}{R}\right)$. Then we have $\|(x_1, y_1)\| > R$, $\|(x_2, y_2)\| > R$, but

$$|f(x_1, y_1) - f(x_2, y_2)| = |e^{-1} - 2\, e^{-2}| = (e - 2)\, e^{-2} > 0. \tag{21}$$

Thus the limit cannot exist.

Question 4. (Continuity) *Let* $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$ *be a linear function. Prove that it is continuous (that is, it is continuous at every point in its domain.)*

Proof. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ be arbitrary. Since $\boldsymbol{f}$ is linear, it is a matrix representation $A = (a_{ij})$. Now we have, for any $\boldsymbol{x} \in \mathbb{R}^N$,

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| &= \|A\,\boldsymbol{x} - A\,\boldsymbol{x}_0\| \\ &= \|A\,(\boldsymbol{x} - \boldsymbol{x}_0)\| \\ &= [(a_{11}\,(x_1 - x_{01}) + \cdots + a_{1N}\,(x_N - x_{0N}))^2 + \cdots + (a_{M1}\,(x_1 - x_{01})^2 + \cdots)^2]^{1/2} \\ &\leqslant [M\,N^2\,(\max|a_{ij}|)^2\,(\max|x_l - x_{0l}|)^2]^{1/2} \\ &\leqslant \sqrt{M}\,N \max|a_{ij}|\,\|\boldsymbol{x} - \boldsymbol{x}_0\|. \end{aligned} \tag{22}$$

Now for any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2\sqrt{M}\,N \max|a_{ij}|}$, we have

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| \leqslant \frac{\varepsilon}{2} < \varepsilon. \tag{23}$$

Therefore $\boldsymbol{f}$ is continuous. □

Question 5. (Open/closed sets) *Let* $A := \{(x, y) \in \mathbb{R}^2 |\, x < y\}$.

a) *Is it open? Is it closed?*

b) *Find its interior.*

c) *Find its closure.*

d) *Find its boundary.*

e) *Find its cluster points.*

Justify all your answers.

Solution.

a) A is open but not closed.

- A is open.

  Take any $(x_0, y_0) \in A$. Take $r = \frac{y_0 - x_0}{2}$. Then for all $(x, y) \in B((x_0, y_0), r)$, we have

$$|x - x_0| < r, |y - y_0| < r \tag{24}$$

  which gives

$$y - x \geqslant (y_0 - x_0) - |x - x_0| - |y - y_0| > 0. \tag{25}$$

  Thus $B((x_0, y_0), r) \subseteq A$.

- A is not closed. We prove $A^c = \{(x, y) \in \mathbb{R}^2 | \, x \geqslant y\}$ is not open.

  Take $(x_0, y_0) = (0, 0) \in A^c$. Then for any $r > 0$, the point $\left(\frac{r}{2}, 0\right) \in B(\mathbf{0}, r)$ but is not a member of A.

b) Since A is open, $A^o = A$.

c) We claim it's closure is $B := \{(x, y) \in \mathbb{R}^2 | \, x \leqslant y\}$. First similar to a) we can prove that B^c is open so B is closed. As $A \subseteq B$, $\overline{A} \subseteq B$ by definition of closure.

Let F be any closed set, $A \subseteq F$, we now prove $B \subseteq F$. Once this is done, we can conclude that $B \subseteq \cap_{A \subseteq F, F \text{ closed}} F = \overline{A}$ and consequently $B = \overline{A}$.

We show $B \subseteq F$ through proving $F^c \subseteq B^c$, that is, if $(x_0, y_0) \in F^c$, then $x_0 > y_0$. Take any $(x_0, y_0) \in F^c$. Then since F^c is open there is $r > 0$ such that

$$B((x_0, y_0), r) \subseteq F^c \subseteq A^c = \{(x, y) \in \mathbb{R}^2 | \, x \geqslant y\}. \tag{26}$$

Now consider $(x, y) = (x_0 - r/2, y_0) \in B((x_0, y_0), r)$. We have

$$x_0 - \frac{r}{2} \geqslant y_0 \Longrightarrow x_0 > y_0 \Longrightarrow (x_0, y_0) \in B^c. \tag{27}$$

Thus the proof ends.

d) The boundary is $\{(x, y) \in \mathbb{R}^2 | \, x = y\}$.

e) The cluster points are $\{(x, y) \in \mathbb{R}^2 | \, x \leqslant y\}$. Take any (x_0, y_0) satisfying $x_0 \leqslant y_0$. Let U be any open set containing (x_0, y_0). Then there is $r > 0$ such that

$$B((x_0, y_0), r) \subseteq U. \tag{28}$$

All we need to show is

$$[B((x_0, y_0), r) - \{(x_0, y_0)\}] \cap A \neq \varnothing \tag{29}$$

or equivalently, there is $(x, y) \in B((x_0, y_0), r)$ different from (x_0, y_0) such that $x < y$. This is easy: Take

$$x = x_0 - r/2, y = y_0. \tag{30}$$

Question 6. (Open/closed sets) *Let* $A \subseteq \mathbb{R}^N$. *Prove* $(\overline{A^c})^c = A^o$.

Proof. We prove through two steps:

1. $(\overline{A^c})^c \subseteq A^o$.

   Since $A^c \subseteq \overline{A^c}$, $(\overline{A^c})^c \subseteq (A^c)^c = A$. Furthermore as $\overline{A^c}$ is closed, $(\overline{A^c})^c$ is open. Now by definition of A^o, $(\overline{A^c})^c \subseteq A^o$.

2. $A^o \subseteq (\overline{A^c})^c$.

Let $\boldsymbol{x} \in A^o$. Then there is an open set U such that $\boldsymbol{x} \in U \subseteq A$. This means $U \cap (A^c) = \varnothing \Longrightarrow A^c \subseteq U^c$. But U^c is closed. Therefore $\overline{A^c} \subseteq U^c$ which means $U \subseteq (\overline{A^c})^c$. Consequently $\boldsymbol{x} \in (\overline{A^c})^c$ and the proof ends. □

Remark. A better way to prove 2. is the following.

$(A^o)^c$ is closed. And since $A^o \subseteq A$, $A^c \subseteq (A^o)^c$. Now by definition of closure we have

$$\overline{A^c} \subseteq (A^o)^c \Longleftrightarrow (\overline{A^c})^c \supseteq A^o. \tag{31}$$