MATH 217 FALL 2013 HOMEWORK 1 SOLUTIONS

- This homework consists of 10 problems of 3 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.

Question 1. *Find a bounded sequence of real numbers that is divergent.*

Discussion. The understanding is that a sequence is convergent if

1. it is bounded, and
2. it is not oscillating.

Therefore we try an oscillating sequence. For example $x_n = (-1)^n$.

Solution. We prove it is not Cauchy: $\exists \varepsilon_0 > 0$, $\forall N \in \mathbb{N}$, $\exists m, n > N$, $|x_m - x_n| \geqslant \varepsilon_0$. Clearly taking any $\varepsilon_0 \leqslant 2$ does the job.

Question 2. *Find a divergent sequence $\{x_n\}$ such that for every $m \in \mathbb{N}$,*

$$\lim_{n \longrightarrow \infty} (x_{n+m} - x_n) = 0. \tag{1}$$

Discussion. The m is in fact a decoy: If we can find a divergent $\{x_n\}$ such that $\lim_{n \longrightarrow \infty} (x_{n+1} - x_n) = 0$, then for any **fixed** m,

$$\lim_{n \longrightarrow \infty} (x_{n+m} - x_n) = \lim_{n \to \infty} [(x_{n+m} - x_{n+m-1}) + \cdots + (x_{n+1} - x_n)] = 0 + \cdots + 0 = 0. \tag{2}$$

Note that there are only a fixed finite number of terms add together.

Solution. Take $x_n = n^a$ for any $0 < a < 1$, or take $x_n = \ln(n)$. For example, for $x_n = n^a$, we have

$$|x_{n+m} - x_n| = a\,\xi^{a-1}(n + m - n) = m\,a\,\xi^{a-1} \tag{3}$$

for some $\xi \in (n, n+m)$ by mean value theorem. When $n \longrightarrow \infty$, $\xi \longrightarrow \infty$ and since $a < 1$, $\xi^{a-1} \longrightarrow 0$. Consequently

$$\lim_{n \longrightarrow \infty} (x_{n+m} - x_n) = 0. \tag{4}$$

Question 3. *Find a function $f: \mathbb{R} \mapsto \mathbb{R}$ that is nowhere continuous, but its absolute value $|f|$ is everywhere continuous.*

Solution. Define

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}. \tag{5}$$

Then $|f(x)| = 1$ for all $x \in \mathbb{R}$ which is obviously continuous.

On the other hand, we can prove that for any $x_0 \in \mathbb{R}$, f is not continuous at x_0. This splits into two cases:

- Case 1. $x_0 \in \mathbb{Q}$. Then there is a sequence of $x_n \notin \mathbb{Q}$ such that $x_n \longrightarrow x_0$. But then $\lim_{x_n \longrightarrow x_0} f(x_n) = -1 \neq f(x_0)$.
- Case 2. $x_0 \notin \mathbb{Q}$. Then there is a sequence of $x_n \in \mathbb{Q}$ such that that $x_n \longrightarrow x_0$. But then $\lim_{x_n \longrightarrow x_0} f(x_n) = 1 \neq f(x_0)$.

Question 4. *Find an infinitely differentiable function* f *such that* $\lim_{x \longrightarrow \infty} f(x) = 0$ *holds but* $\lim_{x \longrightarrow \infty} f'(x) = 0$ *does not hold.*

> **Discussion.** The idea is that f oscillates more and more as $x \longrightarrow \infty$ so that although the amplitude of the oscillation $\longrightarrow 0$, the slope does not. So we take $\sin x$, modulate it through multiplication of $f(x) \longrightarrow 0$, then change its frequency through composition: $\sin(g(x))$ so that when taking derivative, the new factor $g'(x)$ would counter $f(x)$.

Solution. Define

$$f(x) = e^{-x} \sin(e^x). \tag{6}$$

Then since $e^{-x}, e^x, \sin x$ are all infinitely differentiable everywhere, so is f.

We have

$$e^{-x} \leqslant f(x) \leqslant e^{-x} \tag{7}$$

so by Squeeze Theorem $\lim_{x \longrightarrow \infty} f(x) = 0$.

On the other hand,

$$f'(x) = -e^{-x} \sin(e^x) + \cos(e^x). \tag{8}$$

We prove by contradiction. Assume $\lim_{x \longrightarrow \infty} f'(x) = 0$. Then since $\lim_{x \longrightarrow \infty} e^{-x} \sin(e^x) = 0$, we must have $\lim_{x \longrightarrow \infty} \cos(e^x) = 0$. Now take $x_n = \ln(2 n \pi)$. Note that $x_n \longrightarrow \infty$ and $\cos(e^{x_n}) = 1$. Contradiction.

Question 5. *Find a function that is infinitely differentiable (that is* $f^{(n)}$ *exists for all* $n \in \mathbb{N}$*) and satisfy* $f(0) = 1$, $f(x) = 0$ *for all* $|x| \geqslant 1$.

Solution. Consider the function $g(x) = \begin{cases} \exp[-1/x] & x > 0 \\ 0 & x \leqslant 0 \end{cases}$. We prove that it is infintely differentiable. Once this is done, we set

$$f(x) = \frac{g(1-x) \cdot g(x+1)}{g(1)^2}. \tag{9}$$

Then f is infinitely differentiable and $f(0) = 1$, $f(x) = 0$ for all $|x| \geqslant 1$.

To show that $g(x)$ is infinitely differentiable, we prove by induction. Let $Q(n)$ be the statement:

> $g^{(n)}(x)$ exists for all x, and $g^{(n)}(x) = \begin{cases} P_n(1/x) \exp[-1/x] & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ where P_n is a polynomial.

- $Q(1)$. It is clear that $g'(x) = 0$ for $x < 0$ and $g'(x) = \left(\frac{1}{x^2}\right) \exp[-1/x] =: P_1(1/x) \exp[-1/x]$ for $x > 0$. Thus all we need to prove is $g'(0) = 0$.

  We prove through definition: It is easy to see

  $$\lim_{x \longrightarrow 0-} \frac{g(x) - g(0)}{x} = 0. \tag{10}$$

  On the other hand,

  $$\lim_{x \longrightarrow 0+} \frac{g(x) - g(0)}{x} = \lim_{x \longrightarrow 0+} \frac{1}{x} \exp[-1/x] = \lim_{t \longrightarrow +\infty} t\, e^{-t} = 0 \tag{11}$$

  where we have used L'Hospitale: the limit is of the type $\frac{\infty}{\infty}$ so

  $$\lim_{t \longrightarrow +\infty} \frac{t}{e^t} = \lim_{t \longrightarrow +\infty} \frac{1}{e^t} = 0. \tag{12}$$

Thus

$$\lim_{x\longrightarrow 0}\frac{g(x)-g(0)}{x} \tag{13}$$

exists and equals 0.

- $Q(n)\Longrightarrow Q(n+1)$. Assume $Q(n)$:

  $g^{(n)}(x)$ exists for all x, and $g^{(n)}(x)=\begin{cases} P_n(1/x)\exp[-1/x] & x>0\\ 0 & x\leqslant 0\end{cases}$ where P_n is a polynomial.

  Then we clearly have $g^{(n)}$ is differentiable at $x\neq 0$ and takes the values

$$g^{(n+1)}(x)=\begin{cases}\left[-P_n'\left(\frac{1}{x}\right)+P_n\left(\frac{1}{x}\right)\right]\left(\frac{1}{x}\right)^2 e^{-1/x}=:P_{n+1}\left(\frac{1}{x}\right)e^{-1/x} & x>0\\ 0 & x<0\end{cases}. \tag{14}$$

  Thus all we need to show is $g^{(n+1)}(0)$ exists and equals to 0. We have

$$\lim_{x\longrightarrow 0-}\frac{g^{(n)}(x)-g^{(n)}(0)}{x}=0 \tag{15}$$

  and

$$\lim_{x\longrightarrow 0+}\frac{g^{(n)}(x)-g^{(n)}(0)}{x}=\lim_{x\longrightarrow 0+}\frac{1}{x}P_n\left(\frac{1}{x}\right)e^{-1/x}=\lim_{t\longrightarrow +\infty}\frac{t\,P_n(t)}{e^t}. \tag{16}$$

  Since $t\,P_n(t)$ is still a polynomial, application of L'Hospitale finitely many times yields

$$\lim_{t\longrightarrow +\infty}\frac{t\,P_n(t)}{e^t}=\cdots=\lim_{t\longrightarrow +\infty}\frac{1}{e^t}=0. \tag{17}$$

  Thus $Q(n)\Longrightarrow Q(n+1)$ holds.

Question 6. *Find a differentiable function $f:\mathbb{R}\mapsto\mathbb{R}$ such that f' is not continuous.*

Discussion. It should be understood that the following cannot hold: f is differentiable on $(a,b)\ni x_0$, both $\lim_{x\longrightarrow x_0+}f'(x)$ and $\lim_{x\longrightarrow x_0-}f'(x)$ exist but do not equal. Therefore, for f' to be not continuous, the left/right limits must not exist, that is f' must "oscillate". For oscillating functions, we have the Dirichlet function and $\sin(1/x)$. The former is clearly hard to handle so we try the latter.

Solution. Define

$$f(x):=\begin{cases}|x|^a\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases} \tag{18}$$

for some $a\in(1,2]$. Clearly f is differentiable for $x\neq 0$. At 0, we have

$$\lim_{x\longrightarrow 0}\frac{f(x)-f(0)}{x}=\lim_{x\longrightarrow 0}|x|^{a-1}\sin\left(\frac{1}{x}\right)=0 \tag{19}$$

thanks to Squeeze Theorem.

Now when $x>0$,

$$f'(x)=a\,x^{a-1}\sin\left(\frac{1}{x}\right)-x^{a-2}\cos\left(\frac{1}{x}\right). \tag{20}$$

That $\lim_{x \longrightarrow 0} f'(x)$ does not exist can be shown similarly as in the Solution to Question 4.

Question 7. *Find a differentiable function* $f: \mathbb{R} \mapsto \mathbb{R}$ *such that* $f'(0) > 0$ *but* f *is not increasing on any* (a, b) *containing* 0.

Discussion. Recall that, if f' exists and is $\geqslant 0$ on an interval (a, b), then f is increasing. Thus this example shows that $\geqslant 0$ on the whole interval is really necessary.

Solution. We consider

$$f(x) = k\,x + x^2 \sin\left(\frac{1}{x}\right). \tag{21}$$

That this function is differentiable at all x can be proved as in the last problem. We have

$$f'(x) = \begin{cases} k - \cos\left(\frac{1}{x}\right) + 2\,x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ k & x = 0 \end{cases}. \tag{22}$$

Thus $f'(0) > 0$ as long as $k > 0$.

To show that f is not increasing on any interval containing 0, we explore the values of $f(x)$ at $x_n = \frac{1}{2\,n\,\pi + \pi/2}$ and $y_n = \frac{1}{(2\,n-1)\,\pi + \pi/2}$. We have $x_n < y_n$, and

$$f(x_n) - f(y_n) = k\,(x_n - y_n) + x_n^2 + y_n^2 \geqslant 2\,x_n\,y_n - k\,\pi\,x_n\,y_n \tag{23}$$

Thus when $k < 2/\pi$ we have $f(x_n) > f(y_n)$. As $x_n, y_n \longrightarrow 0$, we see that f is not increasing on any (a, b) containing 0.

Remark. A sharper method is the following. We can actually conclude that

If $k \leqslant 1$ then f is not increasing on any (a, b) containing 0; On the other hand, if $k > 1$ then there is a small interval containing 0 such that f is increasing.

- $k \leqslant 1$. All we need to do is to show that there are $a_n < b_n$, $a_n, b_n \longrightarrow 0$ such that $f'(x) < 0$ for $x \in (a_n, b_n)$.

  We have

  $$f'(x) = k - \sqrt{1 + 4\,x^2} \cos\left(\frac{1}{x} + \theta(x)\right) \tag{24}$$

  for $\theta(x)$ satisfying $\tan(\theta) = 2\,x$. Thus $\theta(x)$ is differentiable and $\theta(x) \longrightarrow 0$ as $x \longrightarrow 0$. As $\frac{1}{x} \longrightarrow \infty$ when $x \longrightarrow \infty$, there are $x_n \longrightarrow 0$ such that

  $$\frac{1}{x_n} + \theta(x_n) = 2\,n\,\pi; \tag{25}$$

  Now as

  $$f'(x_n) = k - \sqrt{1 + 4\,x_n^2} < 0 \tag{26}$$

  there is $\delta_n > 0$ such that

  $$f'(x) < 0 \qquad \forall x \in (x_n - \delta_n, x_n + \delta_n) \tag{27}$$

  thanks to the continuity of $f'(x)$ for $x > 0$.

- $k > 1$. In this case set $\delta := \frac{\sqrt{k-1}}{2}$. Then we have, for all $x \in (-\delta, \delta)$,

  $$f'(x) \geqslant k - \sqrt{1 + 4\,x^2} \geqslant k - (1 + 2\,x^2) \geqslant k - (1 + 2\,\delta^2) = \frac{k-1}{2} > 0. \tag{28}$$

Therefore f is increasing in $(-\delta, \delta)$.

Question 8. *Find a function* $f: [0,1] \mapsto \mathbb{R}$ *that is bounded on* $[0,1]$ *but not Riemann integrable.*

Solution. Consider the Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}. \tag{29}$$

It is clearly bounded. To see that it is not Riemann integrable, we check that for any partition $0 = x_0 < x_1 < \cdots < x_n = 1$, we have

$$\max_{x \in [x_{i-1}, x_i]} f = 1, \qquad \min_{x \in [x_{i-1}, x_i]} f = 0. \tag{30}$$

Consequently the upper and lower sums:

$$U(f, P) = 1, \qquad L(f, P) = 0 \tag{31}$$

for all partitions P. This gives $U(f) = 1 \neq 0 = L(f)$ and the function is not Riemann integrable.

Question 9. *Find a function* $f: [0,1] \mapsto \mathbb{R}$ *such that there is* $F: [0,1] \mapsto \mathbb{R}$ *such that* $F' = f$*, but* f *is not Riemann integrable on* $[0,1]$*.*

Note. My intention was to require $F' = f$ on the closed interval [0, 1]. During grading I realized that I didn't make this point clearly enough and many of you find examples with $F' = f$. As this is my fault I decided not to deduct any point in the case. Please contact me if I forgot to do that with your solution.

Discussion. We know that a function is not integrable if at least one of the following happens:

1. f is not bounded;
2. The set $D: = \{x \in [0, 1] | \; f$ is not continuous at $x\}$ does not have Lebesgue measure zero – necessarily D has to contain more points than the set of rationals.

We would like our f' to satisfy one of the above. It is clear that 2. is hard to achieve (not possible though – see remark after solution) so we focus on 1. Furthermore it is clearly easier to start from the construction of the anti-derivative $F(x)$.

Solution. Take $F(x) := \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then as in several previous problems we can show that F is differentiable at all $x \in \mathbb{R}$ and

$$f(x) := F'(x) = \begin{cases} 2\,x \sin \dfrac{1}{x^2} - \dfrac{2}{x} \cos \dfrac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{32}$$

Taking $x_n := \frac{1}{\sqrt{2\,n\,\pi}}$ we see that $f(x)$ is not bounded on $[0,1]$ and therefore is not Riemann integrable.

Remark. David managed to find the following paper:

MR0425042 (54 #13000) Goffman, Casper A bounded derivative which is not Riemann integrable. Amer. Math. Monthly 84 (1977), no. 3, 205–206.

where a bounded function $f(x)$ is constructed that is not Riemann integrable and satisfies

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) \tag{33}$$

for some $F(x)$. Note that this example answers both Questions 8 and 9.

The basic idea is as follows. Let $r_n \in \mathbb{Q} \cap [0,1]$ be all the rational numbers, listed as a countable sequence. Let $\delta \in (0, 1/3)$. Consider

$$O := \cup_{n=1}^{\infty} (r_n - \delta^n, r_n + \delta^n). \tag{34}$$

O is a dense open set. One can prove that any open set in $\mathbb{R}$ can be represented as the union

$$O = \cup_{k=1}^{\infty} I_k \tag{35}$$

where each $I_k := (a_k, b_k)$ is an open interval and $I_k \cap I_l = \varnothing$ whenever $k \neq l$. Now let

$$g(x) := \begin{cases} x+1 & -1 \leqslant x \leqslant 0 \\ 1-x & 0 \leqslant x \leqslant 1 \\ 0 & x \notin [-1,1] \end{cases}. \tag{36}$$

Define

$$g_k(x) := g\left(\frac{x - (a_k + b_k)/2}{a_k - a_k} \right) \tag{37}$$

and then

$$f(x) := \sum_{k=1}^{\infty} g_k(x) = \begin{cases} g_k(x) & x \in (a_k, b_k) \\ 0 & x \notin O \end{cases} \tag{38}$$

is clearly bounded and satisfies $L(f) < U(f)$ so is not Riemann integrable. It should emphasized here that (a_k, b_k) "jumps around" in $[0,1]$ and does not follow the usual "left-to-right" order.

To make $f(x)$ a derivative, we need to slightly modify the construction. Let $J_k := (c_k, d_k) \subset (a_k, b_k)$ such that $\frac{c_k + d_k}{2} = \frac{a_k + b_k}{2}$ and $|c_k - d_k| < |a_k - b_k|^2$. Now re-define

$$g_k(x) := g\left(\frac{x - (c_k + d_k)/2}{d_k - c_k} \right) \tag{39}$$

and still let $f(x) := \sum_{k=1}^{\infty} g_k(x)$. Define $F(x) := \sum_{k=1}^{\infty} G_k(x)$ where $G_k(x) := \int_0^x g_k(t)\, \mathrm{d}t$. Let $x_0 \in [0,1]$. We have the following cases.

1. $x_0 \in O$. Then $x_0 \in I_k$ for some k. For any $x \in I_k$, we have

$$F(x) - F(x_0) = G_k(x) - G_k(x_0) = \int_{x_0}^{x} g_k(t)\, \mathrm{d}t \Longrightarrow F'(x) = g_k(x) = f(x) \tag{40}$$

   thanks to FTC version 2.

2. $x_0 \notin O$. Take any $x \neq x_0$. Wlog assume $x > x_0$. We try to get an upper bound of the size of the set $[x_0, x] \cap (\cup J_n) = \cup([x_0, x] \cap J_n)$ which would be an upper bound of $F(x)$. We only need to consider those J_n such that $[x_0, x] \cap J_n \neq \varnothing$. Fix one such n. We have $[x_0, x] \cap J_n \subseteq J_n$. This gives (we use $|\cdot|$ to denote the size of a set – length in the case of intervals)

$$|[x_0, x] \cap J_n| \leqslant |J_n| < |I_n|^2. \tag{41}$$

   On the other hand, we have

$$|[x_0, x] \cap I_n| \geqslant \frac{1}{2} (|I_n| - |J_n|) > \frac{1}{2} |I_n| (1 - |I_n|) \geqslant \frac{1}{2} |I_n| (1 - |O|) \geqslant \delta |I_n| \tag{42}$$

for some $\delta > 0$. Therefore

$$G_n(x) - G_n(x_0) := \int_{x_0}^{x} g_n(t)\,\mathrm{d}t \leqslant |[x_0, x] \cap J_n| \leqslant C\, |[x_0, x] \cap I_n|^2. \tag{43}$$

This gives

$$|F(x) - F(x_0)| \leqslant C \sum_{n=1}^{\infty} |[x_0, x] \cap I_n|^2 \leqslant C \left(\sum_{n=1}^{\infty} |[x_0, x] \cap I_n| \right)^2 \leqslant C\, (x - x_0)^2. \tag{44}$$

which gives $F'(x_0) = 0$.

Question 10. *Find a function $f: \mathbb{R} \mapsto \mathbb{R}$ that is unbounded on every interval (a, b). Recall that a function is bounded on an interval (a, b) if there is $M > 0$ such that $\forall x \in (a, b)$, $|f(x)| < M$.*

Solution. We modify the Dirichlet function:

$$f(x) := \begin{cases} q & x \in \mathbb{Q}, x = \dfrac{p}{q} \text{ with } q > 0, (p, q) \text{ co-prime} \\ 0 & x \notin \mathbb{Q} \end{cases}. \tag{45}$$

Then for any interval (a, b), we claim there are $r_n \in \mathbb{Q}$ satisfying $r_n \in (a, b)$, $r_n = \frac{p_n}{q_n}$ with p_n, q_n co-prime, and $q_n \longrightarrow \infty$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there are infinitely many rational numbers in (a, b). The difficulty here is to make sure p_n, q_n co-prime and $q_n \longrightarrow \infty$. There are several ways.

- For any $n > -\log_2(b - a) + 1$, we consider the rational numbers $Q_n := \left\{ \frac{2k+1}{2^n} | k \in \mathbb{Z} \right\}$. Clearly $2k + 1$ and 2^n are co-prime. On the other hand as

  $$\frac{2k+1}{2^n} - \frac{2k-1}{2^n} = 2^{-(n-1)} < b - a \tag{46}$$

  there is $r_n \in Q_n \cap (a, b)$. We have $f(r_n) = 2^n$ and the proof ends.
- Several of you have come up with the following beautiful argument: Assume the contrary, that is f is bounded on (a, b) with upper bound M. But there are only finitely many rational numbers of the form p/q with (p, q) co-prime and $q \leqslant M$. Contradiction.

Remark. David (again!) managed to find the following "Conway's Base 13 Function" whose primary purpose is to serve as a function satisfying the intermediately value property but is not continuous on any interval $(a,\, b)$.[1] Note that this function more than settles Question 10: its image on any interval is $\mathbb{R}$, that is, $\forall (a, b) \subseteq \mathbb{R}$, $f((a, b)) = \mathbb{R}$. Details of the construction can be found on wiki.

1. Note that $\sin \frac{1}{x}$ is an example of discontinuous function satisfying IVP. But it is only discontinuous at one single point.