Contents

Note.

- The final is cumulative so please also review material covered before the midterm.
- The exercises and problems in this article does not cover every possible topic in the midterm exam.
- You should also review homework and lecture notes.
- Please try to work on the exercises and problems before looking at the solutions.

F. Higher Order Partial Derivatives

1. Exercises

Exercise 1. Calculate second order partial derivatives for the following function.

$$f(x,y)=(x^2+y^2)^{1/2}. \tag{1}$$

Then generalize your result to $f\colon \mathbb{R}^N \mapsto \mathbb{R}$,

$$f(\boldsymbol{x})=\|\boldsymbol{x}\|. \tag{2}$$

Justify your generalization.

Exercise 2. Let $f(x,y,z)=e^{xyz}$. Calculate

$$\frac{\partial^3 f}{\partial x \partial y \partial z}. \tag{3}$$

Exercise 3. Find $a\in\mathbb{R}$ such that $u(x,\ y,\ z)=e^{ax}\sin y\cos z$ solves the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{4}$$

Exercise 4. Assume the function $u(x,y)$ satisfies

$$5\,x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0. \tag{5}$$

Let $x=e^{\xi}, y=e^{\eta}$ and $v(\xi,\eta)=u(x,y)$. Find the equation satisfied by $v(\xi,\eta)$.

2. Solutions to Exercises

Exercise 1.

We have

$$\frac{\partial f}{\partial x}=\frac{x}{(x^2+y^2)^{1/2}};\ \frac{\partial f}{\partial y}=\frac{y}{(x^2+y^2)^{1/2}}. \tag{6}$$

Taking derivative again we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{3/2}}; \tag{7}$$

$$\frac{\partial^2 f}{\partial x\partial y} = -\frac{x\,y}{(x^2+y^2)^{3/2}}; \tag{8}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}}. \tag{9}$$

Generalization:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}=\begin{cases} -\dfrac{x_i\,x_j}{(x_1^2+\cdots+x_N^2)^{3/2}} & i\neq j\\ \dfrac{(x_1^2+\cdots+x_N^2)-x_i^2}{(x_1^2+\cdots+x_N^2)^{3/2}} & i=j\end{cases}. \tag{10}$$

Exercise 2.

We have

$$\frac{\partial f}{\partial z}=x\,y\,e^{xyz}, \tag{11}$$

$$\frac{\partial^2 f}{\partial y\partial z}=(x+x^2\,y\,z)\,e^{xyz}, \tag{12}$$

$$\frac{\partial^3 f}{\partial x\partial y\partial z}=[1+3\,x\,y\,z+x^2\,y^2\,z^2]\,e^{xyz}. \tag{13}$$

Exercise 3.

We calculate

$$\frac{\partial^2 u}{\partial x^2} = a^2\,e^{ax}\sin y\cos z, \tag{14}$$

$$\frac{\partial^2 u}{\partial y^2} = -e^{ax}\sin y\cos z, \tag{15}$$

$$\frac{\partial^2 u}{\partial z^2} = -e^{ax}\sin y\cos z. \tag{16}$$

So $a=\pm\sqrt{2}$.

Exercise 4.

We have

$$\frac{\partial v}{\partial \xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}=x\frac{\partial u}{\partial x};\ \frac{\partial v}{\partial \eta}=y\frac{\partial u}{\partial y}; \tag{17}$$

$$\frac{\partial^2 v}{\partial \xi^2} = x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}; \tag{18}$$

$$\frac{\partial^2 v}{\partial \xi\partial \eta} = x\,y\frac{\partial^2 u}{\partial x\partial y}; \tag{19}$$

$$\frac{\partial^2 v}{\partial \eta^2} = y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}. \tag{20}$$

So the equation satisfied by v is

$$5\frac{\partial^2 v}{\partial \xi^2}+2\frac{\partial^2 v}{\partial \xi\partial \eta}+\frac{\partial^2 v}{\partial \eta^2}-5\frac{\partial v}{\partial \xi}-\frac{\partial v}{\partial \eta}=0. \tag{21}$$

3. Problems

Problem 1. Let $\Omega\subseteq\mathbb{R}^N$ be open. Let $f(x,y)$ be such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x\partial y}$ exist for $(x,y)\in\Omega$, and furthermore all three functions are continuous at $(x_0,y_0)\in\Omega$. Prove that $\frac{\partial^2 f}{\partial y\partial x}$ exists at (x_0,y_0) and furthermore

$$\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)=\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0). \tag{22}$$

G. Taylor Expansion

1. Exercises

Exercise 5. Let $f(x, y) = y/x$. Find its Taylor polynomial to degree 3 at $(1, 1)$.

Exercise 6. Find the Taylor polynomial to degree $n \in \mathbb{N}$ of $f(x, y, z) = x^2 + y^2 + z^2$ at $(1, 0, 0)$.

Exercise 7. Let $f(x,\ y) = \frac{\cos x}{\cos y}$. Find a second degree polynomial $Q(x, y) = a + b\,x + c\,y + d\,x^2 + e\,x\,y + f\,y^2$ such that

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - Q(x, y)}{(x^2 + y^2)} = 0. \tag{23}$$

Justify.

2. Solutions to exercises

Exercise 5.

We calculate

$$\frac{\partial f}{\partial x} = -y\,x^{-2}, \quad \frac{\partial f}{\partial y} = x^{-1}; \tag{24}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\,y\,x^{-3}, \frac{\partial^2 f}{\partial x \partial y} = -x^{-2}, \frac{\partial^2 f}{\partial y^2} = 0; \tag{25}$$

$$\frac{\partial^3 f}{\partial x^3} = -6\,y\,x^{-4}, \frac{\partial^3 f}{\partial x^2 \partial y} = 2\,x^{-3}, \tag{26}$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = 0,\ \frac{\partial^3 f}{\partial y^3} = 0. \tag{27}$$

At $(1, 1)$ the above become respectively,

$$-1, 1, 2, -1, 0, -6, 2, 0, 0. \tag{28}$$

So the Taylor polynomial of degree 3 is

$$1 - (x-1) + (y-1) + (x-1)^2 - (x-1)\,(y-1) - (x-1)^3 + (x-1)^2\,(y-1). \tag{29}$$

Exercise 6. We calculate, at $(1, 0, 0)$,

$$f(1, 0, 0) = 1; \tag{30}$$

$$\frac{\partial f}{\partial x} = 2\,x = 2; \frac{\partial f}{\partial y} = 2\,y = 0; \frac{\partial f}{\partial z} = 2\,z = 0; \tag{31}$$

$$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 2; \frac{\partial^2 f}{\partial z^2} = 2, \tag{32}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial^2 f}{\partial z \partial x} = 0. \tag{33}$$

It is clear that all higher order partial derivatives are identically 0. Therefore the Taylor polynomial for different n are:

$$P_0(x, y, z) = 1; \tag{34}$$

$$P_1(x, y, z) = 1 + 2\,(x-1); \tag{35}$$

$$P_2(x, y, z) = 1 + 2\,(x-1) + (x-1)^2 + y^2 + z^2. \tag{36}$$

and for all $n \geqslant 2$,

$$P_n(x, y, z) = P_2(x, y, z). \tag{37}$$

Exercise 7. Consider $Q(x,\ y) = P_2(x,\ y)$, the second degree Taylor polynomial of f at $(0,\ 0)$. Then we know

$$f(x, y) = P_2(x, y) + R_2(x, y) \tag{38}$$

with

$$\lim_{(x,y) \to (0,0)} \frac{R_2(x, y)}{x^2 + y^2} = 0. \tag{39}$$

So $P_2(x,\ y)$ satisfies the requirement. Calculation gives

$$P_2(x, y) = 1 - \frac{1}{2}\,(x^2 - y^2). \tag{40}$$

3. Problems

Problem 2. Let $f(x,\ y) \in C^n$ for some $n \in \mathbb{N}$. Let $Q_n(x, y) := \sum_{0 \leqslant i+j \leqslant n} a_{ij}\,x^i\,y^j$. Assume

$$\lim_{(x,y) \longrightarrow (0,0)} \frac{f(x, y) - Q_n(x, y)}{(x^2 + y^2)^{n/2}} = 0. \tag{41}$$

Then

$$a_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0, 0). \tag{42}$$

H. Optimization Theory

1. Exercises

Exercise 8. Is $(0,0)$ a stationary point for the following functions? Is it a local maximizer or minimizer?

$$f_1(x,y) = x^2 - 4\,x\,y + 6\,y^2 - 2; \tag{43}$$

$$f_2(x,y) = (x^2 + y^2)^{1/2}; \tag{44}$$

$$f_3(x,y) = (x+y)^2 - y^2. \tag{45}$$

Exercise 9. Let $f(x,y) = x^4 + 2\,y^2 - 3\,x^2\,y$.

- Find all local minimizers of $f(x,y)$.
- Prove that along every straight line passing the origin, $(0,0)$ minimizes $f(x,y)$.

2. Solutions to exercise

Exercise 8. First check

$$\operatorname{grad} f_1 = \begin{pmatrix} 2\,x - 4\,y \\ -4\,x + 12\,y \end{pmatrix}; \tag{46}$$

$$\operatorname{grad} f_2 = (x^2 + y^2)^{-1/2} \begin{pmatrix} x \\ y \end{pmatrix}; \tag{47}$$

$$\operatorname{grad} f_3 = 2 \begin{pmatrix} x + y \\ x \end{pmatrix}. \tag{48}$$

The formula for f_2 only holds when $(x,y) \neq (0,0)$. At $(0,0)$ it is easy to check that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1 \tag{49}$$

(Note that f is not differentiable at $(0,0)$, but we don't need differentiability to define gradients).

So $(0,0)$ is stationary point for f_1, f_3 but not for f_2.

On the other hand, we have

$$f_1(x,y) = (x - 2\,y)^2 + 2\,y^2 - 2 \geqslant -2 = f_1(0,0) \tag{50}$$

$$f_2(x,y) \geqslant 0 = f_2(0,0) \tag{51}$$

so $(0,0)$ is local minimizer for both f_1, f_2.

For f_3 we have

$$f_3(x,y) = x\,(x + 2\,y). \tag{52}$$

For any $r > 0$, we have $f_3\left(\frac{r}{2}, -\frac{r}{2}\right) < 0 = f_3(0,0)$ and $f_3\left(\frac{r}{2}, \frac{r}{2}\right) > 0 = f_3(0,0)$ so $(0,0)$ is neither local maximizer nor local minimizer.

Exercise 9. We have

$$\operatorname{grad} f = \begin{pmatrix} 4\,x^3 - 6\,x\,y \\ 4\,y - 3\,x^2 \end{pmatrix} \tag{53}$$

Solving $\operatorname{grad} f = \mathbf{0}$ we have

$$x = y = 0. \tag{54}$$

Now the Hessian matrix at $(0,0)$ is $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ which is positive semi-definite so we cannot conclude anything from it.

So instead we factorize

$$f(x,y) = (x^2 - 2\,y)\,(x^2 - y). \tag{55}$$

Now it is easy to show that $(0,0)$ is neither a local maximizer nor a local minimizer.

On the other hand, any line passing $(0,0)$ can be represented by $\left\{ t \begin{pmatrix} u \\ v \end{pmatrix} \middle|\ t \in \mathbb{R} \right\}$. Along this line we have

$$\begin{aligned} f(t\,u, t\,v) &= u^4\,t^4 - 3\,u^2\,v\,t^3 + 2\,v^2\,t^2 \\ &= t^2\,(u^2\,t - v)\,(u^2\,t - 2\,v). \end{aligned} \tag{56}$$

If $u = 0$ or $v = 0$ then clearly $t = 0$ is local minimizer; If $u \neq 0$ we have

$$f(t\,u, t\,v) = u^4\,t^2 \left(t - \frac{v}{u^2}\right)\left(t - \frac{2\,v}{u^2}\right) > 0 \tag{57}$$

if $|t| < \left|\frac{v}{u^2}\right|$.

3. Problems

I. Jordan Measures

1. Exercises

Exercise 10. Let $A := \left\{ \left(\frac{1}{m}, \frac{1}{n} \right) | \, m, n \in \mathbb{N} \right\}$. Prove that $\mu(A) = 0$.

Exercise 11. Let $D(x)$ be the Dirichlet function (1 when $x \in \mathbb{Q}$, 0 when $x \notin \mathbb{Q}$). Let A be its graph over $[0, 1]$:

$$A := \{(x, D(x)) | \, x \in [0, 1]\}. \tag{58}$$

Prove that $\mu(A) = 0$. Thus "graph has measure zero" $\not\Longrightarrow$ function integrable.

Exercise 12. Let $A := \left\{ \left(x, \sin\frac{1}{x} \right) | \, x \in (0, 1) \right\}$. Prove that $\mu(A) = 0$.

Exercise 13. Let $A, B \subseteq \mathbb{R}^N$ be measurable. Assume $\mu(B) = 0$. Prove $\mu(A \cup B) = \mu(A - B) = \mu(A)$.

2. Solutions to exercises

Exercise 10. For any $\varepsilon > 0$, take $N \in \mathbb{N}$ bigger than $2/\varepsilon$. Then take $I_1 := [0, \varepsilon/2] \times [0, 1]$ and $I_2 := [0, 1] \times [0, \varepsilon/2]$. We have

$$\forall m > N, n > N, \quad \left(\frac{1}{m}, \frac{1}{n} \right) \in I_1 \cup I_2 \tag{59}$$

Now set $I_3, \ldots, I_{N^2+2}$ to be the remaining single points. We have

$$A \subseteq \cup_{n=1}^{N^2+1} I_n \tag{60}$$

with $\sum_{n=1}^{N^2+2} \mu(I_n) < \varepsilon$. Therefore $\mu(A) = 0$.

Exercise 11. Take $I_1 := [0, 1] \times \{0\}$ and $I_2 := [0, 1] \times \{1\}$. Then we have $A \subseteq I_1 \cup I_2$ and $\sum_{n=1}^{2} \mu(I_n) = 0$. Therefore $\mu(A) = 0$.

Exercise 12. For any $\varepsilon > 0$, write

$$A := A_1 + A_2 \tag{61}$$

where $A_1 = A \cap [0, \varepsilon] \times \mathbb{R}$ and $A_2 = A \cap [\varepsilon, 1] \times \mathbb{R}$. Then since $\sin \frac{1}{x}$ is continuous on $[\varepsilon, 1]$ we have $\mu(A_2) = 0$. On the other hand clearly $\mu_{\text{out}}(A_1) \leqslant \varepsilon$. Consequently

$$\mu_{\text{out}}(A) \leqslant \varepsilon. \tag{62}$$

The arbitrariness of ε now gives the result.

Exercise 13. We prove $\mu(A \cup B)$. Once this is done, we have

$$\mu(A) = \mu((A \cap B) \cup (A - B)) = \mu(A - B) \tag{63}$$

since $\mu(A \cap B) = 0$.

As $A \cup B$ is measurable, it suffices to prove that $\mu_{\text{out}}(A \cup B) = \mu_{\text{out}}(A)$. Take any $\varepsilon > 0$. Then there is a simple graph $C_1 \supseteq A$ such that $\mu_{\text{out}}(C_1) \leqslant \mu(A) + \varepsilon/2$. On the other hand, as $\mu(B) = 0$, there is a simple graph $C_2 \supseteq B$ such that $\mu(C_2) < \varepsilon/2$. Now we have $A \cup B \subseteq C_1 \cup C_2$ and $\mu(C_1 \cup C_2) \leqslant \mu(C_1) + \mu(C_2) < \mu(A) + \varepsilon$. The arbitrariness of ε now gives the result.

3. Problems

Problem 3. Let W be a collection of (any number of) open Jordan measurable sets. Let

$$E := \cup_{A \in W} A. \tag{64}$$

Prove

$$\mu_{\text{in}}(E) \leqslant \sup_{A_1, \ldots, A_n \in W} \sum_{i=1}^{n} \mu(A_i). \tag{65}$$

Does the conclusion still hold if the "open" assumption is dropped?

Problem 4. Let $A \subseteq \mathbb{R}^N$ and $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$. Assume $\mu(A) = 0$ and $\boldsymbol{f}$ is Holder continuous, that is there are constants $a, C > 0$ such that

$$\forall \boldsymbol{x} \neq \boldsymbol{y}, \quad \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \leqslant C \, \|\boldsymbol{x} - \boldsymbol{y}\|^a. \tag{66}$$

Prove that $\mu(\boldsymbol{f}(A)) = 0$.

J. Theory and Calculation of Riemann Integrals

1. Exericises

Exercise 14. Calculate

$$I := \int_{[0,\pi]^2} \sin^2 x \sin^2 y \, \mathrm{d}(x,y). \tag{67}$$

Exercise 15. (USTC2) Calculate the volume of the intersection of $x^2 + y^2 \leqslant R^2$ and $x^2 + z^2 \leqslant R^2$.

Exercise 16. (USTC2) Calculate

$$I = \int_A \frac{\mathrm{d}(x,y,z)}{(1+x+y+z)^2} \tag{68}$$

where

$$A := \{(x,y,z) | \, x,y,z \geqslant 0, x+y+z \leqslant 1\}, \tag{69}$$

Exercise 17. Let $f(x)$ be continuous on $[a,b]$. Prove that

$$\int_0^a \left[\int_0^x f(x)\, f(y)\, \mathrm{d}y \right] \mathrm{d}x = \frac{1}{2} \left[\int_0^a f(x)\, \mathrm{d}x \right]^2. \tag{70}$$

Exercise 18. (USTC2) Calculate

$$I := \int_A z \, \mathrm{d}(x,y,z) \tag{71}$$

where A is between $x^2+y^2+z^2 = 2\,a\,z$ and $x^2+y^2+z^2 = a\,z$.

2. Solutions to exercises

Exercise 14. By Fubini,

$$I = \int_0^\pi \left[\int_0^\pi \sin^2 x \sin^2 y \, \mathrm{d}y \right] \mathrm{d}x = \frac{\pi^2}{4}.$$

Exercise 15. Denote the intersection by Ω. We have

$$\begin{aligned} \int_\Omega \mathrm{d}(x,y,z) &= 8 \int_{x^2+y^2 \leqslant R^2, x,y \geqslant 0} \sqrt{R^2 - x^2} \, \mathrm{d}(x,y) \\ &= 8 \int_0^R \sqrt{R^2 - x^2} \left[\int_0^{\sqrt{R^2-x^2}} \mathrm{d}y \right] \mathrm{d}x \\ &= \frac{16}{3} R^3. \end{aligned} \tag{72}$$

Exercise 16. Let

$$D := \{(x,y) | \, (x,y,z) \in A\}. \tag{73}$$

Then $D = \{(x,y) | \, x+y \leqslant 1, x,y \geqslant 0\}$. Thus

$$\begin{aligned} I &= \int_D \left[\int_0^{1-x-y} \frac{\mathrm{d}z}{(1+x+y+z)^2} \right] \mathrm{d}(x,y) \\ &= \cdots \\ &= \frac{3}{4} - \ln 2. \end{aligned} \tag{74}$$

Exercise 17. Switch the order of the integration.

Exercise 18. Apply spherical coordinates, we have

$$T^{-1}(A) = \left\{ (r, \varphi, \psi) | \; a \cos\psi \leqslant r \leqslant 2\, a \cos\psi, 0 \leqslant \varphi \leqslant 2\pi, 0 \leqslant \psi \leqslant \frac{\pi}{2} \right\}. \tag{75}$$

Thus

$$\begin{aligned} I &= \int_{T^{-1}(A)} r^3 \cos\psi \sin\psi \, \mathrm{d}(r,\varphi,\psi) \\ &= \cdots \\ &= \frac{5}{4} \pi a^4. \end{aligned} \tag{76}$$

3. Problems

Problem 5. Let $A := \{(x,y) | \, x^2 + y^2 \leqslant 1\}$. Consider approximating $I := \int_A \sin(x+y) \, \mathrm{d}(x,y)$ by

$$I_h := \sum_{(ih, jh) \in A} \sin(i\,h, j\,h)\, h^2. \tag{77}$$

For what h can we guarantee that

$$|I - I_h| < 0.001? \tag{78}$$

Problem 6. Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be bounded. Let $A \subseteq \mathbb{R}^N$ be Jordan measurable. Prove that f is integrable on A if and only if for every bounded function $g: \mathbb{R}^N \mapsto \mathbb{R}^N$,

$$U(f+g, A) = U(f, A) + U(g, A). \tag{79}$$

Problem 7. Let $f: [\alpha, \beta] \mapsto \mathbb{R}$ be continuous. Let (r, θ) be polar coordinates. Let

$$D_f := \{(r,\theta) | \, 0 \leqslant r \leqslant f(\theta), \alpha \leqslant \theta \leqslant \beta\}. \tag{80}$$

Prove that the area of D_f is

$$\frac{1}{2} \int_\alpha^\beta f^2(\theta)\, \mathrm{d}\theta. \tag{81}$$

Problem 8. Switch the order of integration in

$$\int_{-\pi/2}^{\pi/2} \left[\int_0^{\cos\theta} f(r,\theta)\, \mathrm{d}r \right] \mathrm{d}\theta \tag{82}$$

where (r, θ) is Polar coordinates.

K. Numbers

1. Exercises

Exercise 19. Let $F := \{r + s\sqrt{2} | r, s \in \mathbb{Q}\}$ be equipped with the usual addition and multiplication. Prove that F is a field.

Exercise 20. For the above F, define relations

$$r_1 + s_1\sqrt{2} <_A r_2 + s_2\sqrt{2} \Leftrightarrow (r_1 - r_2) + (s_1 - s_2)\sqrt{2} < 0 \tag{83}$$

and

$$r_1 + s_1\sqrt{2} <_B r_2 + s_2\sqrt{2} \Leftrightarrow (r_1 - r_2) - (s_1 - s_2)\sqrt{2} < 0 \tag{84}$$

Prove that $<_A, <_B$ both make F an ordered field. Denote it by F_A, F_B.

2. Solutions to Exercises

Exercise 19. We first check the axioms of addition:

- $(r_1 + s_1\sqrt{2}) + (r_2 + s_2\sqrt{2}) = (r_1 + r_2) + (s_1 + s_2)\sqrt{2} \in F$;
- $(r_1 + s_1\sqrt{2}) + (r_2 + s_2\sqrt{2}) = (r_1 + r_2) + (s_1 + s_2)\sqrt{2} = (r_2 + s_2\sqrt{2}) + (r_1 + s_1\sqrt{2})$.
- Associativity is similar;
- The element 0 is $0 + 0\sqrt{2}$.
- $-(r + s\sqrt{2}) = (-r) + (-s)\sqrt{2}$;

Next check the axioms of multiplication. That $x\,y \in F$, $x\,y = y\,x$, $x\,(y\,z) = (x\,y)\,z$ are obvious. The element 1 is $1 + 0\sqrt{2}$. The only thing we need to check is, if $r + s\sqrt{2} \neq 0$ then

$$\frac{1}{r + s\sqrt{2}} \in F. \tag{85}$$

We have

$$\frac{1}{r + s\sqrt{2}} = \frac{r}{r^2 - 2s^2} + \frac{-s}{r^2 - 2s^2}\sqrt{2} \in F. \tag{86}$$

Note that $r^2 - 2s^2 \neq 0$ for all $r, s \in \mathbb{Q}$.

Finally the distributive law is obviously true.

Exercise 20. $<_A$ part is trivial.

We check that $<_B$ is an order. That any $x, y \in F$ exactly one of the three relations holds is obvious. Now assume

$$r_1 + s_1\sqrt{2} <_B r_2 + s_2\sqrt{2} <_B r_3 + s_3\sqrt{2}. \tag{87}$$

Then we have

$$(r_1 - r_2) + (s_2 - s_1)\sqrt{2} < 0 \tag{88}$$

and

$$(r_2 - r_3) + (s_3 - s_2)\sqrt{2} < 0. \tag{89}$$

Add the two inequalities together we see that $r_1 + s_1\sqrt{2} <_B r_3 + s_3\sqrt{2}$.

It is obvious that this order is consistent with addition. Now take $r_1 + s_1\sqrt{2} >_B 0$ and $r_2 + s_2\sqrt{2} >_B 0$. By definition this means

$$r_1 - s_1\sqrt{2} > 0, r_2 - s_2\sqrt{2} > 0. \tag{90}$$

Now we calculate their product to be

$$r_1 r_2 + 2 s_1 s_2 + (r_1 s_2 + r_2 s_1)\sqrt{2}. \tag{91}$$

We see that it $>_B 0$ as

$$r_1 r_2 + 2 s_1 s_2 - (r_1 s_2 + r_2 s_1)\sqrt{2} = (r_1 - s_1\sqrt{2})(r_2 - s_2\sqrt{2}). \tag{92}$$

3. Problems

Problem 9. Do the ordered fields F_A, F_B satisfy the LUB property? Justify.

Solutions to Problems

PROBLEM 1. We use f_x, f_y, f_{xy} to denote the partial derivatives.

At any (x_0, y_0) We calculate for any (x, y),

$$\begin{aligned} & \frac{f_x(x_0, y) - f_x(x_0, y_0)}{y - y_0} \\ = & \frac{[f_x(x_0, y) - f_x(x_0, y_0)]\,(x - x_0)}{(x - x_0)\,(y - y_0)} \\ = & \frac{[f(x, y) - f(x, y_0)] - [f(x_0, y) - f(x_0, y_0)]}{(x - x_0)\,(y - y_0)} \\ & + \frac{[f_x(x_0, y) - f_x(x_0, y_0)] - [f_x(\xi, y) - f_x(\xi, y_0)]}{(y - y_0)} \end{aligned}$$

By continuity of f_x, the second term tends to 0 as $(x, y) \to (x_0, y_0)$.

Now similar method as in the lecture notes we can prove that the first term tends to $f_{xy}(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$.

PROBLEM 2. Since $f(x, y) \in C^n$, it has Taylor expansion

$$f(x, y) = P_n(x, y) + R_n(x, y) \tag{93}$$

where $P_n(x, y) = \sum_{0 \leqslant i + j \leqslant n} b_{ij}\, x^i\, y^j$ with

$$b_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0, 0) \tag{94}$$

and

$$\lim_{(x, y) \longrightarrow (0, 0)} \frac{R_n(x, y)}{(x^2 + y^2)^{n/2}} = 0. \tag{95}$$

Thus all we need to prove is that

$$\lim_{(x, y) \longrightarrow (0, 0)} \frac{P_n(x, y) - Q_n(x, y)}{(x^2 + y^2)^{n/2}} = 0 \tag{96}$$

then $P_n(x) = Q_n(x)$. Equivalently, all we need to prove is that if a polynomial $H_n(x)$ of degree n satisfies

$$\lim_{(x, y) \longrightarrow (0, 0)} \frac{H_n(x, y)}{(x^2 + y^2)^{n/2}} = 0 \tag{97}$$

then $H_n(x, y) = 0$.

Let n_0 be the smallest non-negative integer such that there is a term $h_{ij}\ x^i\ y^j$ in $H_n(x, y)$ with $i + j = n_0$ and $h_{ij} \neq 0$.

Now set $x = t$, $y = u\ t$ for $u \in \mathbb{R}$ and let $t \to 0$, we have

$$\lim_{t \to 0} \frac{\left(\sum_{i + j = n_0} h_{ij}\, u^j\right) t^{n_0} + f(t, u)\, t^{n_0 + 1}}{t^n} = 0 \tag{98}$$

where $f(t)$ is a polynomial of t. We see that it must be $\sum_{i + j = n_0} h_{ij}\, u^j = 0$ for all $u \in \mathbb{R}$.

That is

$$h_{n_0, 0} + h_{n_0 - 1, 1}\, u + \cdots + h_{0, n_0}\, u^{n_0} = 0 \tag{99}$$

for all $u \in \mathbb{R}$. Setting $u = 0$ we have $h_{n_0, 0} = 0$. Taking $\frac{\mathrm{d}}{\mathrm{d}u}$ and set $u = 0$ we have $h_{n_0 - 1, 1} = 0$. Keep doing this we have $h_{ij} = 0$ for all $i + j = n_0$. This contradicts the assumption that n_0 is the smallest with some $h_{ij} \neq 0$ with $i + j = n_0$.

PROBLEM 3. For any $\varepsilon > 0$, let B be the simple graph satisfying

$$B \subseteq E^o, \qquad \mu(B) \geqslant \mu_{\text{in}}(E) - \varepsilon. \tag{100}$$

Then since B is compact and $B \subseteq E = \cup_{A \in W} A$, there is a finite subcover:

$$B \subseteq \cup_{i=1}^n A_i. \tag{101}$$

This means

$$\mu(B) \leqslant \sum_{i=1}^n \mu(A_i) \leqslant \sup_{A_1, \ldots, A_n \in W} \sum_{i=1}^n \mu(A_i) \tag{102}$$

therefore

$$\mu_{\text{in}}(E) - \varepsilon \leqslant \sup_{A_1, \ldots, A_n \in W} \sum_{i=1}^n \mu(A_i). \tag{103}$$

The conclusion follows from the arbitrariness of ε.

The conclusion does not hold anymore if we drop the "open" assumption. For example

$$[0, 1] = \cup_{x \in [0, 1]} \{x\} \tag{104}$$

but $\mu(\{x\}) = 0$ for each x.

Problem 4. Note that the statement is wrong. Check "devil's staircase" on wiki to see a Holder continuous function (with $a = \frac{\log 2}{\log 3}$) that maps the Cantor set (try prove that its Jordan measure is 0!) to the unit interval.

The statement is only true when $a = 1$. In this case consider covering A by intervals of the form $I_h := [i_1\, h, (i_1 + 1)\, h] \times \cdots \times [i_N\, h, (i_N + 1)\, h]$. We know that

$$\lim_{h \to 0} n(h)\, h^N = \mu(A) = 0 \tag{105}$$

where $n(h)$ is the number of intervals needed to cover A. But now that $f(I_h) \subseteq$ a ball of radius $\sqrt{2}\ C\ h$ and consequently a cube of side $2\sqrt{2}\, C\, h$ and therefore

$$\mu(f(A)) \leqslant n(h)\, \left(2\sqrt{2}\, C\right)^N h^N \longrightarrow 0. \tag{106}$$

Problem 5. Consider $I_{ij} := [i\,h, (i+1)\,h] \times [j\,h, (j+1)\,h]$. Then there are three cases: $I_{ij} \subseteq A^o$, $I_{ij} \cap \partial A \neq \varnothing$, $I_{ij} \cap A = \varnothing$. We say $(i,j) \in M_1, M_2, M_3$ respectively. We have

$$\left| I - \sum_{(i,j)\in M_1} \int_{I_{ij}} \sin(x+y)\,\mathrm{d}(x,y) \right| \leqslant \sum_{(i,j)\in M_2} \mu(I_{ij}) = \sum_{(i,j)\in M_2} h^2. \tag{107}$$

On the other hand we have similar inequality for $\left| I_h - \sum_{(i,j)\in M_1} \sin(i\,h + j\,h)\,h^2 \right|$. Therefore

$$|I - I_h| \leqslant \sum_{(i,j)\in M_1} \left| \int_{I_{ij}} \sin(x+y)\,\mathrm{d}(x,y) - \sin(i\,h + j\,h)\,h^2 \right| + 2\sum_{(i,j)\in M_2} h^2. \tag{108}$$

Now we have

$$\begin{aligned} & \left| \int_{I_{ij}} \sin(x+y)\,\mathrm{d}(x,y) - \sin(i\,h+j\,h)\,h^2 \right| \\ = & \left| \int_{I_{ij}} |\sin(x+y) - \sin(i\,h+j\,h)| \mathrm{d}(x,y) \right| \\ \leqslant & \int_{I_{ij}} |\sin(x+y) - \sin(i\,h+j\,h)|\,\mathrm{d}(x,y) \\ \leqslant & \ h^2 \max_{(x,y)\in I_{ij}} \|(x,y) - (i\,h, j\,h)\| \\ < & \ 2\,h^3. \end{aligned} \tag{109}$$

Note that there can be no more than $\left(\frac{2}{h}\right)^2$ intervals in M_1, therefore

$$|I - I_h| \leqslant 8\,h + 2\sum_{(i,j)\in M_2} h^2. \tag{110}$$

Now note that if $I_{ij} \cap \partial A \neq \varnothing$, then $I_{ij} \subseteq A_{2h} := \{(x,y)\in\mathbb{R}^2 | \operatorname{dist}((x,y),\partial A) < 2\,h\}$. Thus

$$\sum_{(i,j)\in M_2} h^2 < \mu(A_{2h}) = \pi\,(1+2\,h)^2 - \pi(1-2\,h)^2 = 8\,\pi\,h. \tag{111}$$

Summarizing, we have

$$|I - I_h| < (8 + 16\,\pi)\,h < 100\,h. \tag{112}$$

Thus taking $h < 10^{-5}$ would guarantee what we need.

Problem 6.

- If f is Riemann integrable.

Let $F_n \geqslant f$, $G_n \geqslant g$ be two sequences of simple functions such that

$$\lim_{n\to\infty} \int_A F_n = U(f, A) \tag{113}$$

$$\lim_{n\to\infty} \int_A G_n = U(g, A). \tag{114}$$

Then clearly $F_n + G_n \geqslant f + g$ are also simple functions and thus

$$U(f+g, A) \leqslant \int_A F_n + G_n = \int_A F_n + \int_A G_n. \tag{115}$$

Taking limit $n \to \infty$ now gives

$$U(f+g, A) \leqslant U(f, A) + U(g, A). \tag{116}$$

Note that this holds for all functions, integrable or not.

Now we prove the other direction. Take any simple function $H(\boldsymbol{x}) \geqslant f + g$ and any simple function $F(\boldsymbol{x}) \leqslant f$. Then $H - F \geqslant g$ is a simple function and by definition

$$\int_A H - F \geqslant U(g, A). \tag{117}$$

Consequently we have

$$\begin{aligned} \int_A H &= \int_A F + \int_A H - F \\ &\geqslant \int_A F + U(g, A). \end{aligned} \tag{118}$$

Taking supreme over F we have

$$\int_A H \geqslant L(f, A) + U(g, A) \tag{119}$$

then taking infimum over H we finally reach

$$U(f+g, A) \geqslant L(f, A) + U(g, A). \tag{120}$$

Since f is integrable, $L(f, A) = U(f, A)$ and the conclusion follows.

- Assume

$$U(f+g, A) = U(f, A) + U(g, A) \tag{121}$$

holds for all bounded function g. Take $g=-f$. We have

$$\begin{aligned} 0 &= U(f+g,A) \\ &= U(f,A)+U(-f,A) \\ &= U(f,A)-L(f,A) \end{aligned} \tag{122}$$

and integrability of f follows.

Problem 7. We have

$$\int_\alpha^\beta \left[\int_0^{f(\theta)} r\,\mathrm{d}r\right]\mathrm{d}\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\,\mathrm{d}\theta. \tag{123}$$

Problem 8. The answer is

$$\int_0^1 \left[\int_{-\arccos r}^{\arccos r} f(r,\theta)\,\mathrm{d}\theta\right]\mathrm{d}r. \tag{124}$$

Problem 9.

- $<_A$. Since $<_A$ coincide with the usual order on $\mathbb{R}$, all we need to show is that F is dense in $\mathbb{R}$ yet $F \neq \mathbb{R}$. Since $\mathbb{Q} \subset F$, F is dense in $\mathbb{R}$. Now we prove that $\sqrt{3} \notin F$.

  Assume the contrary. Then there are $r, s \in \mathbb{Q}$ such that $r + s\sqrt{2} = \sqrt{3}$. Taking square we have $(r^2 + 2s^2 - 3) + 2rs\sqrt{2} = 0$, contradicting $\sqrt{2} \notin \mathbb{Q}$.

- $<_B$. Consider the set $E := \left\{-t\sqrt{2} \mid t \in \mathbb{Q}, t < \sqrt{\frac{3}{2}}\right\}$. Obviously E is bounded above and not empty. Assume that

  $$\sup E = r + s\sqrt{2}. \tag{125}$$

  Then we have, for any $t > \sqrt{3/2}$, $-t\sqrt{2} \leqslant_B r + s\sqrt{2}$ which means $r + (s+t)\sqrt{2} \geqslant_B 0$ which by definition is $r - (s+t)\sqrt{2} \geqslant 0$ or

  $$r \geqslant (s+t)\sqrt{2}. \tag{126}$$

  Clearly $=$ cannot hold. Thus we have

  $$r > (s+t)\sqrt{2}. \tag{127}$$

  But then there must be $r' < r$ such that $r' > (s+t)\sqrt{2}$. Thus $r' + s\sqrt{2}$ is an upper bound for E with order $<_B$. But $r' + s\sqrt{2} <_B r + s\sqrt{2}$. Contradiction.