Properties of real numbers

$\mathbb{R}$ is an ordered field

Definition 1. (Field) *A set F is called a field if there are two functions defined: $\oplus, \odot: F \times F \mapsto F$, satisfying the following:*

- *Axioms for addition:*

  i. $x \in F, y \in F \Longrightarrow x \oplus y \in F$;

  ii. $x \oplus y = y \oplus x$;

  iii. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

  iv. *There is an element* 0 *satisfying* $0 \oplus x = x$ *for any* $a \in F$;

  v. *For each* $x \in F$, *there is an element* $y \in F$ *such that* $y \oplus x = 0$. *Denote it by* $-x$.

- *Axioms for multiplication:*

  a) $x \in F, y \in F \Longrightarrow x \odot y \in F$;

  b) $x \odot y = y \odot x$;

  c) $x \odot (y \odot z) = (x \odot y) \odot z$;

  d) *There is an element* $i \in F$ *such that* $i \odot x = x$ *for every* $x \in F$. *Denote it by* 1;

  e) *For every* $x \in F$, *there is a* $y \in F$ *such that* $x \odot y = 1$. *Denote* y *by* x^{-1}.

- *The distributive law:*

  A) *For every* $x, y, z \in F$, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Exercise 1. Prove the uniqueness of the special elements $0, 1$. Also prove that for each $x \in F$ and each $y \in F, y \neq 0$, $-x$ and y^{-1} are unique.

Exercise 2. Give a reasonable definition to the new function $F \times F \times F \mapsto F$ which we hope to denote by $x \oplus y \oplus z$ and justify your definition.

Exercise 3. Let A be a set. Let $W := \{\text{subsets of } A\}$. Define addition and multiplication on W as

$$x \oplus y := x \cup y; \qquad x \odot y := x \cap y. \tag{1}$$

Does this make W a field? Justify your answer.

Remark 2. Clearly we can further define "subtration" and "division" through

$$x - y := x + (-y); \qquad x / y := x\,(y^{-1}). \tag{2}$$

Example 3. Prove that $4 = 2 \oplus 2$.

Proof. By definition,

$$4 = 3 \oplus 1 = (2 \oplus 1) \oplus 1 = 2 \oplus (1 \oplus 1) = 2 \oplus 2. \tag{3}$$

Thus ends the proof. □

Exercise 4. Let $x \in F$. Prove that $x \oplus x \oplus x = 3 \odot x$.

Exercise 5. Prove

a) $x \oplus y = x \oplus z \Longrightarrow y = z$;

b) $x \odot y = x \odot z \Longrightarrow y = z$ unless $x = 0$.

Exercise 6. Denote $x \odot x$ by x^2. Prove that $(x \oplus y)^2 = x^2 \oplus (2 \odot x \odot y) \oplus y^2$.

Exercise 7. Prove the following

a) $x \odot 0 = 0$;

b) $x \odot (-y) = -(x \odot y)$;

c) $(-x) \odot (-y) = x \odot y$;

d) If $x \neq 0$, $(-x)^{-1} = -(x^{-1})$;

Notation. *From now on we will discard* $\odot$ *and* $\oplus$*, and simply use the usual notations* $x \cdot y$ $(x\,y)$, x/y, $x \pm y$.

Definition 4. (Order) *Let S be a set. An "order" on S is a relation, denoted by $<$, with the following two properties:*

i. If $x \in S, y \in S$ then exactly one of the following is true.

$$x < y, x = y, x > y; \tag{4}$$

ii. If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark 5. $\geqslant, \leqslant$ can be defined in the natural way.

Definition 6. (Ordered field) *F is an ordered field if*

i. It is a field;

ii. It has an order;

iii. The field operations are consistent with the order structure:

- $x, y, z \in F, y < z \Longrightarrow x + y < x + z;$
- $x, y \in F, x > 0, y > 0 \Longrightarrow x\,y > 0.$

Exercise 8. Let F be an ordered field. Let $x, y \in F$. Prove that if $x\,y < 0$, then one is positive and the other negative.

Exercise 9. Let F be an ordered field. Let $x, y, z \in F$. Then

a) If $x > 0$ then $-x < 0$ and vice verse;

b) If $x \neq 0$, then $x^2 > 0$; In particular $1 > 0$;

c) If $x > 0$, $y < z$, then $x\,y < x\,z$;

d) If $0 > x > y$, then $0 > \frac{1}{y} > \frac{1}{x}$.

Theorem 7. *$\mathbb{R}$ as constructed in the previous sections is an ordered field.*

$\mathbb{R}$ has least upper bound property

Definition 8. (Upper bound) *Suppose S is an ordered set, and $E \subseteq S$. If there is a $\beta \in S$ such that $x \leqslant \beta$ for every $x \in E$, we say E is bounded above, and called β an upper bound of E.*

Remark 9. Lower bound can be defined similarly.

Definition 10. (Least upper bound) *Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose that there exists an $\alpha \in S$ such that*

i. α is an upper bound of E;

ii. If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the least upper bound (also called "supreme") of E. Denoted as

$$\alpha = \sup E. \tag{5}$$

Remark 11. Greatest lower bound (or infimum) $\alpha = \inf E$ can be defined similarly.

Definition 12. (LUB property) *An ordered set S is said to have the least-upper-bound (LUB) property if:*

$$E \subseteq S, E \neq \varnothing, E \text{ is bounded above}, \Longrightarrow \sup E \text{ exists in } S. \tag{6}$$

Exercise 10. Prove that $\mathbb{Q}$ does not have LUB property.

Exercise 11. S has least-upper-bound property $\Longleftrightarrow$ S has greatest lower bound property.

Theorem 13. *Let F be an ordered field with LUB property. Then for every $x > 0$, there is a unique $y > 0$ such that $y^3 = x$.*

Proof. That y is unique is trivial since $y_1 < y_2 \Longrightarrow y_1^3 < y_2^3$.

To show existence, consider $A := \{t \in F |\, t \geqslant 0, t^3 < x\}$ and set $y := \sup A$. Note that since $0^3 = 0 < x$, we have $0 \in A$ and therefore y exists. Furthermore taking $z = \min\left\{1, \frac{x}{2}\right\}$ we have

$$z^3 \leqslant z \leqslant \frac{x}{2} < x \Longrightarrow z \in A \Longrightarrow y > 0. \tag{7}$$

Now we show that it cannot hold that $y^3 < x$ or $y^3 > x$. Assume the contrary. Two cases.

- $y^3 < x$. Then $1 < y^{-3}\,x$. If we can find $\varepsilon > 0$ such that $(1+\varepsilon)^3 < y^{-3}\,x$ then $[(1+\varepsilon)\,y]^3 < x$ and we have a contradiction.

Thus it suffices to show that $1 < x \Longrightarrow \exists \varepsilon > 0, (1+\varepsilon)^3 < x$. We have

$$(1+\varepsilon)^3 = 1 + 3\,\varepsilon + 3\,\varepsilon^2 + \varepsilon^3 = 1 + (3 + 3\,\varepsilon + \varepsilon^2)\,\varepsilon. \tag{8}$$

Now take $\varepsilon = \min\left\{1, \frac{x-1}{8}\right\}$. We have

$$(1+\varepsilon)^3 = 1 + (3 + 3\,\varepsilon + \varepsilon^2)\,\varepsilon \leqslant 1 + 7\,\varepsilon \leqslant 1 + \frac{7}{8}\,(x-1) < x. \tag{9}$$

Thus we are done.

- $y^3 > x$. In this case all we need is $(1-\varepsilon)^3 > y^{-3}\,x$. In light of

$$(1-\varepsilon)^3 = 1 - 3\,\varepsilon + 3\,\varepsilon^2 - \varepsilon^3 > 1 - (3 + \varepsilon^2)\,\varepsilon \tag{10}$$

the proof is similar to that for the previous case.

Thus $y^3 = x$ and the proof ends. □

Exercise 12. Prove that $y_1 < y_2 \Longrightarrow y_1^3 < y_2^3$ with no assumption on the signs of y_1, y_2.

Exercise 13. Fill in the details for the $y^3 > x$ case.

Exercise 14. Let F be an ordered field with LUB property. Let $\alpha \in \mathbb{Q}$. Define α-th power in the natural way. Then for every $x > 0$, there is a unique $y > 0$ such that $y^\alpha = x$.

Problem 1. Let F be an ordered field with LUB property. Let $\alpha \in \mathbb{R}$. Define x^α for all $x \in F, x > 0$. (Hint: See Problem 6 of Chapter 1 in **(Baby Rudin)**)

Problem 2. Let F be an ordered field with LUB property. Fix $b > 1, y > 0$. Prove that there is a unique $x \in \mathbb{R}$ such that $b^x = y$. (Hint: See Problem 7 of Chapter 1 in **(Baby Rudin)**)

Theorem 14. $\mathbb{R}$ *as constructed in the previous sections has the LUB property.*

Proof. We consider the following cases:

- All the upper bounds are positive. Then $E \cup \mathbb{R}^+$ is not empty. We identify real numbers with cuts and define

$$\alpha := \cup_{\xi \in E \cup \mathbb{R}^+} \xi. \tag{11}$$

- 0 is an upper bound but there is no negative upper bounds. In this case by definition $\sup E = 0 \in \mathbb{R}$.

- There is at least one negative upper bound. Define

$$F := \{-\alpha |\ \alpha \text{ is an upper bound for } E,\ \alpha < 0\}. \tag{12}$$

Now treat member of F as cuts and define $\eta := -\xi$ where

$$\xi := \cup_{\alpha \in F} \alpha. \tag{13}$$

Clearly $\eta = \sup E$. □

Problem 3. Let F be an ordered field satisfying LUB. Prove that there is $x \in F$ such that $x^2 = 2$.

Exercise 15. Let F be an ordered field satisfying LUB. Let $a \in F$ be $a > 0$. Prove that there is $x \in F$ such that $x^2 = a$.

Archimedean

Definition 15. *A ordered field F is said to be Archimedean if and only if $\mathbb{N}$ does not have an upper bound in F. Here $\mathbb{N}$ is defined as $\{1, 1+1, 1+1+1, \ldots\}$.*

Remark 16. It is obvious that $\mathbb{R}$ is Archimedean.

Theorem 17. *A ordered field F satisfying LUB then Archimedian.*

Proof. Assume $\mathbb{N}$ is bounded from above. Then there is least upper bound $a = \sup \mathbb{N}$. By definition of sup , $a - 1$ is not a upper bound for $\mathbb{N}$. Thus there is $n \in \mathbb{N}$ such that $n > a - 1$. But then $a < n + 1 \in \mathbb{N}$. Contradiction. □

Exercise 16. Find an ordered field that is Archimedean but does not satisfy LUB.

Theorem 18. *An ordered field F is Archimedean $\Longleftrightarrow$ $\mathbb{Q}$ is dense in F.*

Proof.

- $\Longrightarrow$. Take any $x, y \in F, x < y$. We prove that there is $z \in \mathbb{Q}$ such that $x < z < y$. It is clear that it suffices to discuss the situation $0 < x < y$. In this case we need to find $m, n \in \mathbb{N}$ such that

$$x < \frac{m}{n} < y \Longleftrightarrow n\,x < m < n\,y. \tag{14}$$

  Since F is Archimedean, there is $n \in \mathbb{N}$ such that $n\,(y - x) > 1$. Fix this n. Consider the set $A := \{k \in \mathbb{N} |\, k < n\,y\}$. Again since F is Archimedean, this set is finite and we take $m = \max A$. We claim that $m > n\,x$. Assume otherwise, then $m < n\,x \Longrightarrow m + 1 < n\,x + n\,(y - x) = n\,y$. We see that $m + 1 \in A$. Contradiction.

- $\Longleftarrow$. Take any positive $y \in F$. We show that it cannot be an upper bound for $\mathbb{N}$.

  Since $\mathbb{Q}$ is dense in F, there is $\frac{m}{n}$ such that

$$0 < \frac{m}{n} < \frac{1}{y} \Longrightarrow m\,y < n \Longrightarrow y < n. \tag{15}$$

  Thus F is Archimedean. □

$\mathbb{R}$ is unique

Theorem 19. *$\mathbb{R}$ is the unique ordered Archimedean field. Or equivalently $\mathbb{R}$ is the unique ordered field where $\mathbb{Q}$ is dense.*